Translational and Rotational Invariance in Networked Dynamical Systems

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***Abstract*—In this paper, we study the translational and rotational ($SE(N)$) invariance properties of locally interacting multi-agent systems. We focus on a class of networked dynamical systems, in which the agents have local pairwise interactions, and the overall effect of the interaction on each agent is the sum of the interactions with other agents. We show that such systems are $SE(N)$-invariant if and only if they have a special, *quasi-linear* form. The $SE(N)$-invariance property, sometimes referred to as left invariance, is central to a large class of kinematic and robotic systems. When satisfied, it ensures independence to global reference frames. In an alternate interpretation, it allows for integration of dynamics and computation of control laws in the agents' own reference frames. Such a property is essential in a large spectrum of applications, e.g., navigation in GPS-denied environments. Because of the simplicity of the quasi-linear form, this result can impact ongoing research on design of local interaction laws. It also gives a quick test to check if a given networked system is $SE(N)$-invariant.**

***Index Terms*—translational and rotational invariance, networked systems, pairwise interaction.**

I. Introduction

In this paper we present necessary and sufficient conditions for a multi-agent system with pairwise interactions to be invariant under translations and rotations of the inertial frame in which the dynamics are expressed (i.e. $SE(N)$-invariant). This kind of invariance is important because it allows agents to execute their control laws in their body reference frame [1], [2], [3], using information measured in their body reference frame, without effecting the global evolution of the system. This is critical for any scenario where global information about an agent's reference frame is not readily available, for example coordinating agents underground, underwater, inside of buildings, in space, or in any GPS denied environment [4], [5], [6].

We assume that the agents are kinematic in N-dimensional Euclidean space, and their control laws are computed as the sum over all neighbors of pairwise interactions with the neighbors. We prove that the dynamics are $SE(N)$-invariant if and only if the pairwise interactions are *quasi-linear*, meaning linear in the difference between the states of the two agents, multiplied by a scalar gain which depends only on the distance between the states of the two agents. This result can be used as a test (does a given multi-agent controller require global information?), or as a design specification (a multi-agent controller is required that uses only local information, hence only quasi-linear pairwise interactions can be considered). It can also be used to test hypothesis about local interaction laws in biological (e.g., locally interacting cells) and physical systems.

We prove the result for agents embedded in Euclidean space of any dimension, and the result holds for arbitrary graph topologies, including directed or undirected, switching, time varying, and connected or unconnected. We show that many existing multi-agent protocols are quasi-linear and thus $SE(N)$-invariant. Examples include the interactions from the classical n-body problem [7] and most of the existing multi-agent consensus and formation control algorithms, e.g., [8], [9], [10], [11], [12], [13], [14]. In particular, explicit consensus algorithms implemented using local information in the agents' body frames [6] satisfy the $SE(N)$-invariance property, as expected. We also show that some multi-agent interaction algorithms, such as [15], are not $SE(N)$-invariant, and therefore cannot be implemented locally in practice. We also consider a sub-class of $SE(N)$-invariant (and therefore quasi-linear) pairwise interaction systems, and show that they reach a consensus, using the graph Laplacian to represent the system dynamics and the typical LaSalle's invariance analysis to show convergence. Finally, we present extensions of the $SE(N)$-invariance notion to discrete-time systems, dynamical systems of higher order and systems with switching topologies. Moreover, for a sub-class of discrete-time $SE(N)$-invariant pairwise interaction systems, we show that they reach consensus by exploiting the quasi-linear structure given by the main result.

With a few exceptions [16], [17], [18], [6], the problem of invariance to global reference frames was overlooked in the multi-agent control and consensus literature. In [16], the authors discuss invariance for the particular cases of $SE(2)$ and $SE(3)$ actions, and with particular focus on virtual structures. Rotational and translational invariance are also discussed for a particular class of algorithms driving a team of agents to a rigid structure in [17]. The problem of invariance to group actions in multi-agent systems was very recently studied in [18], where the authors present a general framework to find all symmetries in a given second-order planar system. The authors' main motivation is to determine changes of coordinates transformations that align the system with the

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symmetry directions and thus aid in stability analysis using LaSalle's principle. This paper is complementary to our work, in the sense that the authors start from a system and find invariants, while in our case we start from an invariance property and find all systems satisfying it. Our results hold for any (finite) dimensional agent state space. Finally, our characterization of invariance is algebraic, and as a result does not require any smoothness assumptions on the functions modeling the interactions. As a result, it can be used for a large class of systems, including discrete-time systems.

Preliminary results from this work were presented in a conference version [19]. The present paper expands on [19] by including all proofs of the main results, as well as new results on the stability of $SE(N)$-invariant systems, switching network topologies, and discrete-time systems. We also provide several new examples with simulations.

The rest of the paper is organized as follows. Section II describes $SE(N)$-invariance from a geometrical point of view. Section III defines necessary concepts and states the main result. The main result is proved in Sections IV, and V. Section VI considers convergence to consensus in a special class of systems. Section VII presents some extensions of $SE(N)$-invariance to discrete-time, higher order systems and systems with switching topologies. Section VIII analyzes the $SE(N)$-invariance of several well-known systems, and conclusions are given in Section IX.

II. Significance of $SE(N)$-invariance

In this section we present $SE(N)$-invariance from a geometrical point of view and give two interpretations which prove to be useful for networked agent systems. Formal definitions will be provided in Sec. III together with the main result of the paper.

$SE(N)$ is the Special Euclidean group that acts on $\mathbb{R}^N$, i.e., the set of all possible rotations and displacements in $\mathbb{R}^N$. As mentioned before, $SE(N)$-invariance is a property related to reference frames. Consider a global inertial (world) reference frame $\{\mathbb{W}\}$, which we call world frame, and another (mobile) reference frame $\{\mathbb{M}\}$, which is related to $\{\mathbb{W}\}$ by the rotation and translation pair $(R, w) \in SE(N)$. Also, consider a networked system with n agents which interact with each other in a pairwise manner, i.e. communication is point-to-point and may be one-way. Let $x_i^{\mathbb{W}}$ and $x_i^{\mathbb{M}}$ be the state of agent i in reference frames $\{\mathbb{W}\}$ and $\{\mathbb{M}\}$, respectively. (See Fig. 1(a) for an illustration in the case of N=3). The states of agent i in the two references frames are related by $x_i^{\mathbb{W}} = Rx_i^{\mathbb{M}} + w$.

In order to relate the velocities of an agent i in the two reference frames, careful consideration must taken about how the velocities are measured and represented. Consider the velocities ${}^{\mathbb{W}}v_i^{\mathbb{W}}$ and ${}^{\mathbb{W}}v_i^{\mathbb{M}}$ measured with respect to the world frame $\{\mathbb{W}\}$ and represented in $\{\mathbb{W}\}$ and $\{\mathbb{M}\}$, respectively. Then these are related by ${}^{\mathbb{W}}v_i^{\mathbb{W}} = R^{\mathbb{W}}v_i^{\mathbb{M}}$. On the other hand, the dynamics of agent i is given by ${}^{\mathbb{W}}v_i^{\mathbb{W}} = f_{ij}(x_i^{\mathbb{W}}, x_j^{\mathbb{W}})$, where we assumed for simplicity that agent i communicates only with agent j.

The notion of $SE(N)$-invariance says that the dynamics of agent i must be the same in all reference frames, i.e. ${}^{\mathbb{W}}v_i^{\mathbb{M}} =$

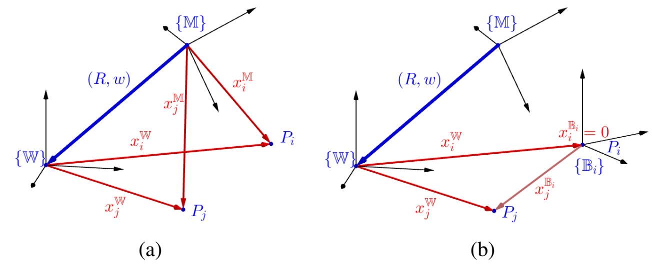


Fig. 1. The diagram in (a) shows the world frame $\{\mathbb{W}\}$, the reference frame $\{\mathbb{M}\}$, two agents i and j and their states in these two frames. The diagram in (b) shows the agents' states expressed in the body frame of agent i.

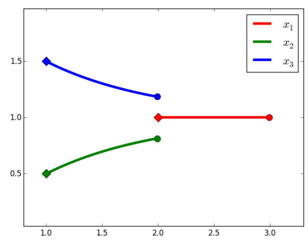
$f_{ij}(x_i^{\mathbb{M}}, x_j^{\mathbb{M}})$ must hold for all $\{\mathbb{M}\}$. A quick substitution yields $R^{\mathbb{W}}v_i^{\mathbb{M}} = f_{ij}(Rx_i^{\mathbb{M}} + w, Rx_j^{\mathbb{M}} + w)$. On the other hand we have $R^{\mathbb{W}}v_i^{\mathbb{M}} = Rf_{ij}(x_i^{\mathbb{M}}, x_j^{\mathbb{M}})$, which implies that SE(N) invariance reduces to $Rf_{ij}(x_i^{\mathbb{M}}, x_j^{\mathbb{M}}) = f_{ij}(Rx_i^{\mathbb{M}} + w, Rx_j^{\mathbb{M}} + w)$ for all values of the states $x_i^{\mathbb{M}}, x_j^{\mathbb{M}}$ and all transformations $(R, w) \in SE(N)$. This is the notion of left invariance that we will define formally in Sec. III. Notice that $SE(N)$-invariance is a basic assumption very common in physical models (i.e. the laws of physics must be the same in all inertial reference frames). In the context of differential geometry, this intuition is formalized and generalized by the notion of left-invariance of vector fields.

In the context of networked systems, each agent maintains an individual mobile reference frame. If the reference frames of all agents coincide, then they achieve global localization (this may be implemented using GPS, SLAM, etc.). However, if we desire a truly distributed system, then the agents must maintain local reference frames, which are not synchronized with each other, and be able to compute their own individual control laws in their own local frames. A special choice of a mobile reference frame is the body frame of an agent. Each agent i is associated with its body frame $\{\mathbb{B}_i\}$, (see Fig. 1(b)). The agents measure (using an on-board sensor such as a camera) and express the states of all their neighbors in their own individual reference frames $\{\mathbb{B}_i\}$. Consequently, if the system is $SE(N)$-invariant, then the agents can compute their individual control laws (their velocities) in their own body frames, without the need of a predefined global reference. Therefore, we consider that, in practice, $SE(N)$-invariance is a very important property of distributed networked systems.

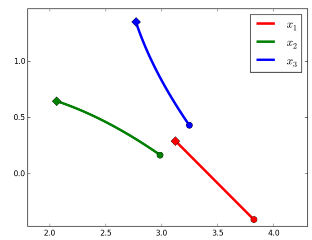
Another interpretation of $SE(N)$-invariance is is related to the networked system's behavior, i.e. the agents' trajectories. The invariance property implies that the system produces the same trajectories in any two reference frames. The trajectories of an agent have the same shape and scale (they are isometric) and are related by the transformation between the two reference frames. Fig. 2 shows an example of trajectories generated by an $SE(2)$-invariant system and one set of trajectories from a non-$SE(2)$-invariant system in two reference frames, respectively.

III. Definitions and Main Result

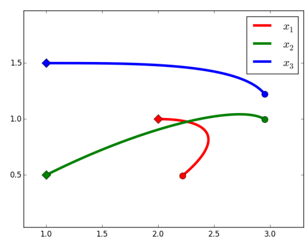
In this section, we introduce the notions and definitions used throughout the paper. The main result of the paper is stated at



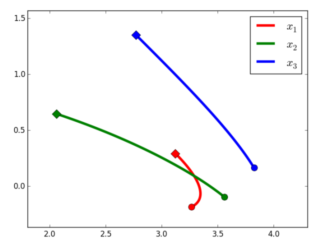
(a) Trajectories of a $SE(2)$-invariant system as seen from $\{\mathbb{W}_1\}$



(b) Trajectories of a $SE(2)$-invariant system as seen from $\{\mathbb{W}_2\}$



(c) Trajectories of a non-$SE(2)$-invariant system as seen from $\{\mathbb{W}_1\}$



(d) Trajectories of a non-$SE(2)$-invariant system as seen from $\{\mathbb{W}_2\}$

Fig. 2. The figure shows the trajectories of two systems in two reference frames $\{\mathbb{W}_1\}$ and $\{\mathbb{W}_2\}$, which are related by a rotation $R(\pi/4)$ in clockwise direction and a translation $w = [1, 1]^T$. Clearly, the trajectories generated by the $SE(2)$-invariant system have the same shape and are related by (R, w), (a) and (b). The shape of the trajectories generated by the non-$SE(2)$-invariant system are different in the two reference frames, (c) and (d).

the end of the section.

For a set S, we use $|S|$ to denote its cardinality. The sets $\mathbb{R}_{\geq a}$ and $\mathbb{Z}_{\geq p}$ represent the interval $[a, \infty)$ and $\{p, p+1, \ldots\}$, where $a \in \mathbb{R}$ and $p \in \mathbb{Z}$, respectively. The notation $\triangleq$ denotes a definition. The canonical basis for the Euclidean space of dimension N, denoted by $\mathbb{R}^N$, is $\{e_1, \ldots, e_N\}$. We use I_N and $\mathbf{1}_N$ to denote the $N \times N$ identity matrix and the $N \times 1$ vector of ones, respectively. The special orthogonal group acting on $\mathbb{R}^N$ is denoted by $SO(N)$. Similarly, $SE(N)$ represents the special Euclidean group of rotations and translations acting on $\mathbb{R}^N$. Throughout the paper, the norm $\|\cdot\|$ refers to the Euclidean norm. The Kronecker product of two matrices is denoted by $\otimes$.

Given a directed graph G, we use $V(G)$ and $E(G) \subseteq V(G) \times V(G)$ to denote its sets of nodes and edges, respectively. An edge $(i, j) \in E(G)$ is interpreted as starting from i and ending at j. An edge starting at i is called an outgoing edge of i, while an edge ending at i is called an incoming edge of i. Given a node $i \in V(G)$, $\mathcal{N}_i^{\rightarrow}$ stands for the set of outgoing neighbors of i, i.e. $\mathcal{N}_i^{\rightarrow} = \{j \in V(G) | (i, j) \in E(G)\}$. Similarly, $\mathcal{N}_i^{\leftarrow} = \{j \in V(G) | (j, i) \in E(G)\}$ represents the set of incoming neighbors of i.

Definition III.1 ($SE(N)$-invariant function)**.** *A function $f : \mathbb{R}^N \times \cdots \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is said to be $SE(N)$-invariant if for all $R \in SO(N)$ and all $x_1, , x_p, w \in \mathbb{R}^N$ the following condition holds:*

$$Rf(x_1, \ldots, x_p) = f(Rx_1 + w, \ldots, Rx_p + w). \tag{1}$$

Definition III.2 (Pairwise Interaction System)**.** *A continuous-time pairwise interaction system is a double (G, F), where G is a graph and $F = \{f_{ij} \mid f_{ij} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N, (i, j) \in E(G)\}$ is a set of functions associated to its edges. Each $i \in V(G)$ labels an agent, and a directed edge (i, j) indicates that node i requests and receives information from node j. The dynamics of each agent are described by*

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j), \tag{2}$$

where f_{ij} defines the influence (interaction) of j on i.

For each agent $i \in V(G)$, we denote the *total interaction* on agent i by

$$S_i(x_1, \ldots, x_{|V(G)|}) = \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j). \tag{3}$$

Definition III.3 ($SE(N)$-Invariance)**.** *A pairwise interaction system (G, F) is said to be $SE(N)$-invariant if, for all $i \in V(G)$, the total interaction functions S_i are $SE(N)$-invariant.*

Definition III.4 (Quasi-linear function)**.** *A function $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is said to be quasi-linear if there is a function $k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that $f(x) = k(\|x\|)x$, for all $x \in \mathbb{R}^{\mathrm{N}}$.*

Definition III.5 (Quasi-linear interaction system)**.** *A pairwise interaction system (G, F) is said to be quasi-linear if the total interaction S_i of each agent i is a sum of quasi-linear functions. Formally, for all $i \in V(G)$:*

$$S_i = \sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}(\|x_j - x_i\|)(x_j - x_i), \tag{4}$$

where $k_{ij} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ are scalar gain functions for all $j \in \mathcal{N}_i^{\rightarrow}$ and $N \geq 3$.

Remark III.6. *The definition of quasi-linearity for pairwise interaction systems is a statement about the overall dynamics of agents. Specifically, Def. III.5 does not imply that the pairwise interaction functions f_{ij} are themselves quasi-linear functions. See Ex. VIII.1 and Remark III.8.*

The main result of this paper can be stated as follows:

Theorem III.7. *A pairwise interaction system (G, F) is $SE(N)$-invariant if and only if it is quasi-linear.*

Remark III.8. *The pairwise interaction form of the systems considered in this paper is a fundamental assumption needed to obtain the main result, Thm. III.7. To illustrate this, consider a system with three agents and the following total interaction function of agent 1, which captures a three-way interaction among agents:*

$$S_1(x_1, x_2, x_3) = \|x_2 - x_1\| (x_3 - x_2).$$

By Def. III.1 S_1 is $SE(N)$-invariant. Indeed,

$$\begin{aligned} &RS_1(x_1, x_2, x_3) = \\ &= \|x_2 - x_1\| R(x_3 - x_2) \\ &= \|Rx_2 + w - (Rx_1 + w)\| (Rx_3 + w - (Rx_1 + w)) \\ &= S_1(Rx_1 + w, Rx_2 + w, Rx_3 + w) \end{aligned}$$

for all $(R, w) \in SE(N)$. However, S_1 cannot be written as a sum of quasi-linear functions.

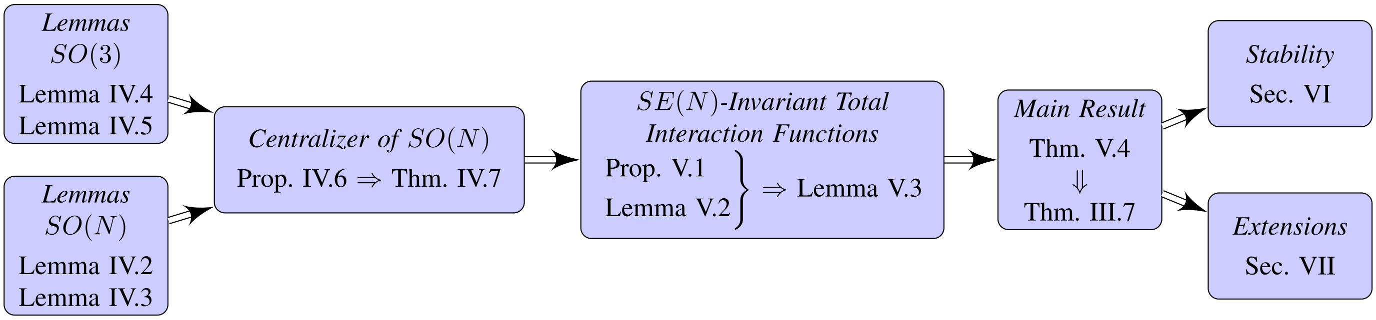


Fig. 3. Diagram of results

Remark III.9. *Since $SE(N)$-invariance is a property of reference frames, it does not imply anything about the stability of the system. The converse does not hold either. Therefore, we can have unstable $SE(N)$-invariant systems and stable systems which are not $SE(N)$-invariant. More details are included in Sec. VIII.*

Remark III.10. *Note that we do not impose any restrictions on the graph G or the set of functions F. The results hold even if G is disconnected and the local interaction functions are not related to each other.*

The main result of the paper (Thm. III.7) can be regarded as a characterization of $SE(N)$-invariant functions arising from pairwise interaction systems. We establish the structure of these $SE(N)$-invariant functions in Sec. IV, where we show that all local interaction functions corresponding to an agent are quasi-linear functions with an additional affine term. We also show that the sum of all affine terms over the neighbors of an agent must vanish. Thus, it follows that the total interaction functions are quasi-linear, i.e. these can be written as sums of quasi-linear functions. As an intermediate step to establishing the form of $SE(N)$-invariant total interaction functions, we prove that functions which commute with $SO(N)$ are quasi-linear. We provide stability results on $SE(N)$-invariant systems in Sec. VI. Finally, in Sec. VII, we include extensions of Thm. III.7 to discrete-time systems, higher order systems and switching topologies. An overview of how the results in the paper follow from each other is presented in Fig. 3.

IV. Characterizing the Centralizers of $SO(N)$

In this section, we prove that functions which commute with $SO(N)$ are quasi-linear, which generalizes the well-known result for linear functions [20]. We establish the general case using induction on $N \geq 3$. The case $N = 2$ is treated separately in App. X.

Let $T = \{f : \mathbb{R}^N \to \mathbb{R}^N\}$ be the set of all transformations acting on $\mathbb{R}^N$. T is a monoid with respect to function composition and is called the *transformation monoid*.

Definition IV.1 (Centralizer)**.** *Let A be a sub-semigroup of T. The* centralizer *(or* commuter*) of A with respect to T is denoted by $C_T(A)$ and is the set of all elements of T that commute with all elements of A, i.e. $C_T(A) = \{f \in T | fg = gf, \forall g \in A\}$.*

The centralizer $C_T(A)$ is a submonoid of T and can be interpreted as the set of transformations invariant with respect to all transformations in A. In other words, the action of $f \in C_T(A)$ on $\mathbb{R}^N$ and then transformed by $g \in A$ is the same as the action of f on the transformed space $g(\mathbb{R}^N)$.

Note that the set of all quasi-linear functions is a submonoid of T, which will be denoted by $QL(N)$.

Before we proceed, we provide two lemmas that are used in subsequent proofs. The following lemma, whose proof is straightforward and omitted, shows the intuitive fact that the only vector invariant under all rotations is the null vector.

Lemma IV.2. *Let $x \in \mathbb{R}^N$. If $Rx = x$ for all $R \in SO(N)$, $N \geq 2$, then $x = 0$.*

Lemma IV.3. *Let $f = (f_1, \ldots, f_N) : \mathbb{R}^N \to \mathbb{R}^N$ such that f commutes with all elements of $SO(N)$. Then $x^T f(x) = \|x\| f_1(\|x\| e_1)$, for all $x \in \mathbb{R}^N$.*

Proof. Let $x \in \mathbb{R}^N$ and $R \in SO(N)$ such that $Rx = \|x\| e_1$ or equivalently $x = R^T \|x\| e_1$. It follows that $f(x) = f(R^T \|x\| e_1) = R^T f(\|x\| e_1)$. Finally, $x^T f(x) = x^T R^T f(\|x\| e_1) = (Rx)^T f(\|x\| e_1) = \|x\| e_1^T f(\|x\| e_1) = \|x\| f_1(\|x\| e_1)$. ☐

The following three lemmas establish the case $N = 3$ which forms the base case of the induction argument used in the proof of Thm. IV.7.

Lemma IV.4. *Let $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ such that $\|u\| = 1$ and $u \neq \pm e_1$. Then* $R_u = \begin{bmatrix} u_1 & 0 & -\sqrt{u_2^2+u_3^2} \\ u_2 & \frac{u_3}{\sqrt{u_2^2+u_3^2}} & \frac{u_1 u_2}{\sqrt{u_2^2+u_3^2}} \\ u_3 & -\frac{u_2}{\sqrt{u_2^2+u_3^2}} & \frac{u_1 u_3}{\sqrt{u_2^2+u_3^2}} \end{bmatrix}$ *is a rotation matrix in $SO(3)$.*

Proof. The matrix satisfies $R_u R_u^T = I_3$ and $\det(R_u) = 1$, and thus it is a rotation matrix in $SO(3)$. ☐

Lemma IV.5. *Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ such that f commutes with all elements of $SO(3)$, then*

$$f_1(x) = -f_1(-x_1, -x_2, x_3) \quad (5)$$
$$f_1(x) = -f_1(-x_1, x_2, -x_3) \quad (6)$$
$$f_2(x) = f_1(x_2, -x_1, x_3) \quad (7)$$
$$f_3(x) = f_1(x_3, x_2, -x_1), \quad (8)$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Proof. Let $R_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ and $R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. The constrains are obtained using the commutation condition $f(Rx) = Rf(x)$ and some algebraic manipulation. From $R_1 f(x) = f(R_1 x)$, we get:

$$\begin{aligned} f_2(x) &= f_1(x_2, -x_1, x_3) && (9) \\ -f_1(x) &= f_2(x_2, -x_1, x_3) = f_1(-x_1, -x_2, x_3) && (10) \\ f_3(x) &= f_3(x_2, -x_1, x_3). && (11) \end{aligned}$$

Similarly, we obtain for R_2:

$$\begin{aligned} f_3(x) &= f_1(x_3, x_2, -x_1) && (12) \\ f_2(x) &= f_2(x_3, x_2, -x_1) && (13) \\ -f_1(x) &= f_3(x_3, x_2, -x_1) = f_1(-x_1, x_2, -x_3) && (14) \end{aligned}$$

and for R_3

$$\begin{aligned} f_1(x) &= f_1(x_1, x_3, -x_2) && (15) \\ f_3(x) &= f_2(x_1, x_3, -x_2) && (16) \\ -f_2(x) &= f_3(x_1, x_3, -x_2). && (17) \end{aligned}$$

Using Eq. (9) and (12) to express f_2 and f_3 in terms of f_1, respectively, and rearranging the variables we obtain the desired constrains.

$$\begin{aligned} f_1(x) &= -f_1(-x_1, -x_2, x_3) \\ f_1(x) &= -f_1(-x_1, x_2, -x_3) \\ f_1(x) &= f_1(x_1, -x_3, x_2) \\ f_1(x) &= -f_1(x_1, x_3, x_2) \\ f_2(x) &= f_1(x_2, -x_1, x_3) \\ f_3(x) &= f_1(x_3, x_2, -x_1). \end{aligned}$$

□

Proposition IV.6. *The centralizer of $SO(3)$ with respect to T is the monoid of quasi-linear functions $QL(3)$.*

Proof. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x \neq \alpha e_1$, $\alpha \in \mathbb{R}$ and $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$. Let $u = \frac{x}{\|x\|}$ and R_u as in Lemma IV.4, we have $x = R_u \|x\| e_1$ and $u_i = \frac{x_i}{\|x\|}$. Using the commutation property we obtain $f(x) = f(R_u \|x\| e_1) = R_u f(\|x\| e_1)$ and writing the equation for f_1, it follows that

$$f_1(x) = u_1 f_1(\|x\| e_1) - \sqrt{u_2^2 + u_3^2}\, f_3(\|x\| e_1). \quad (18)$$

Using the equality from Lemma IV.5, Eq. (8), we have $f_3(\|x\|, 0, 0) = f_1(0, 0, -\|x\|)$. On the other hand, it follows from Eq. (5) that $f_1(0, 0, \alpha) = -f_1(0, 0, \alpha)$, which implies $f_1(0, 0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$. It follows that $f_3(\|x\| e_1) = 0$ for all $x \in \mathbb{R}^3$, $x \neq \alpha e_1$ and $\alpha \in \mathbb{R}$. Thus, Eq. (18) can be simplified to

$$f_1(x) = x_1 \frac{f_1(\|x\| e_1)}{\|x\|} = x_1 k(\|x\|), \quad (19)$$

where $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is $k(\alpha) \triangleq \frac{f_1(\alpha e_1)}{\alpha}$, $\alpha \geq 0$. The general form of $f(x) = k(\|x\|)x$ is obtained using Eq. (7) and (8).

The case $x = 0$ follows easily from Lemma IV.2, because it implies $f(0) = 0$. The remaining case $x = \alpha e_1$ is trivial; $f(\alpha e_1) = [f_1(\alpha e_1)\ f_2(\alpha e_1)\ f_3(\alpha e_1)]^T = [\alpha \frac{f_1(\alpha e_1)}{\alpha}\ 0\ 0]^T = k(\|x\|)x$, where $f_2(\alpha e_1) = 0$ and $f_3(\alpha e_1) = 0$ follow from Eq. (7), (6) and Eq. (8), (5), respectively.

Conversely, if $f \in QL(N)$, then $Rf(x) = R(k(\|x\|)x) = k(\|Rx\|)Rx = f(Rx)$, where $R \in SO(3)$. Thus, we have $f \in C_T(SO(3))$, which concludes the proof. □

Theorem IV.7. *The centralizer of $SO(N)$ with respect to T is the monoid of quasi-linear functions $QL(N)$, for all $N \geq 3$.*

Proof. The proof follows from an induction argument with respect to N. The base case is established by Prop. IV.6. To simplify the notation, given a vector $x = (x_1, \ldots, x_N)$ we will denote by $x_{i:j}$, $i < j$, the sliced vector $(x_i, \ldots, x_j) \in \mathbb{R}^{j-i+1}$.

The induction step: Let $x \in \mathbb{R}^{N+1}$, $x \neq 0$, and $R_1 = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$, where $R \in SO(N)$. Using R_1, it follows that $Rf_{1:N}(x_{1:N}, x_{N+1}) = f_{1:N}(Rx_{1:N}, x_{N+1})$. Applying the induction hypothesis, we obtain

$$f_{1:N}(x_{1:N}, x_{N+1}) = k_1(\|x_{1:N}\|, x_{N+1}) x_{1:N}\,. \quad (20)$$

Similarly, using R_2 we have $Rf_{2:N+1}(x_1, x_{2:N+1}) = f_{2:N+1}(x_1, Rx_{2:N+1})$ and obtain

$$f_{2:N+1}(x_1, x_{2:N+1}) = k_2(\|x_{2:N+1}\|, x_1) x_{2:N+1}\,. \quad (21)$$

Equating Eq. (20) and (21) for f_2 and assuming w.l.o.g. $x_2 \neq 0$, we get a constraint between the two gains

$$k_2(\|x_{2:N+1}\|, x_1) = k_1(\|x_{1:N}\|, x_{N+1})\,. \quad (22)$$

Thus, we obtain f_{N+1} in terms of the gain k_1 by using the last equality from Eq. (21) and (22) to substitute k_2 for k_1

$$f_{N+1}(x_1, \ldots, x_{N+1}) = k_1(\|x_{1:N}\|, x_{N+1}) x_{N+1}\,. \quad (23)$$

Finally, putting all the components of f obtained from Eq. (20) and (23) together and left multiplying it by x^T, we get

$$\begin{aligned} x^T f(x) &= \sum_{i=1}^{N+1} x_i^2 k_1(\|x_{1:N}\|, x_{N+1}) \\ &= \|x\|^2 k_1(\|x_{1:N}\|, x_{N+1}) = \|x\| f_1(\|x\| e_1), \end{aligned}$$

where the last equality follows from Lemma IV.3. It follows that $k_1(\|x_{1:N}\|, x_{N+1}) = \frac{f_1(\|x\| e_1)}{\|x\|} \triangleq k(\|x\|)$. Thus, $f(x) = k(\|x\|)x$ or equivalently $f \in C_T(SO(N))$.

Conversely, we have $QL(N) \subseteq C_T(SO(N))$ (see proof of Prop. IV.6). □

V. $SE(N)$-INVARIANT FUNCTIONS

In this section, we use the result from the previous section that $C_T(SO(N)) = QL(N)$ in order to characterize $SE(N)$-invariant functions that arise from pairwise interaction systems.

Proposition V.1. *A function $h(x_1, x_2) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is $SE(N)$-invariant if and only if h is quasi-linear in $x_2 - x_1$.*

Proof. Trivially, a quasi-linear function $h(x_1, x_2) = k(\|x_2 - x_1\|)(x_2 - x_1)$ is $SE(N)$-invariant. Conversely, if $R = I_N$ and $w = -x_2$, then $h(x_1, x_2) = h(x_1 - x_2, x_2 - x_2) = h(x_1 - x_2, 0) \triangleq \hat{h}(x_2 - x_1)$. Let $x \in \mathbb{R}^N$ and $R \in SO(N)$, it follows that $R\hat{h}(x) = Rh(-x, 0) = h(-Rx, 0) = \hat{h}(Rx)$. Since $\hat{h}$ commutes with all elements of $SO(N)$ it follows that it is quasi-linear. Thus, we have $h(x_1, x_2) = \hat{h}(x_2 - x_1) = k(\|x_2 - x_1\|)(x_2 - x_1)$. □

Lemma V.2. *Let $h_1, h_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$. Then $S(x_0, x_1, x_2) = h_1(x_0, x_1) + h_2(x_0, x_2)$ is an $SE(N)$-invariant function if and only if there exists $k_1(\cdot)$ and $k_2(\cdot)$ such that for all $x_0, x_1, x_2 \in \mathbb{R}^N$ we have*

$$h_1(x_0, x_1) = h_1(x_0, x_0) + k_1(\|x_1 - x_0\|)(x_1 - x_0) \quad (24)$$

$$h_2(x_0, x_2) = h_2(x_0, x_0) + k_2(\|x_2 - x_0\|)(x_2 - x_0) \quad (25)$$

and $h_1(x_0, x_0) + h_2(x_0, x_0) = 0$.

Proof. It is easy to show that if S is the sum of functions satisfying Eq. (24), (25) and the zero-sum constraint, then S is $SE(N)$-invariant. Conversely, let $f_1(a, b) = h_1(a, b) + h_2(a, a)$ and $f_2(a, b) = h_1(a, a) + h_2(a, b)$, where $f_1, f_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $a, b \in \mathbb{R}^N$. It follows immediately that f_1 and f_2 are $SE(N)$-invariant, because $h_1(x_0, x_1) + h_2(x_0, x_2)$ is $SE(N)$-invariant. Therefore, we have by Prop. V.1 that $f_1(a, b) = k_1(\|b - a\|)(b - a)$ and $f_2(a, b) = k_2(\|b - a\|)(b - a)$. Choosing $a = b$ in any of the previous two equations, we obtain $h_1(a, a) + h_2(a, a) = 0$. Finally, we obtain $h_1(a, b) = -h_2(a, a) + f_1(a, b) = h_1(a, a) + k_1(\|b - a\|)(b - a)$ and $h_2(a, b) = -h_1(a, a) + f_2(a, b) = h_2(a, a) + k_2(\|b - a\|)(b - a)$. □

Lemma V.3. *Let $h_1, \ldots, h_p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $p \in \mathbb{Z}_{\geq 2}$. Then $S(x_0, \ldots, x_p) = \sum_{i=1}^p h_i(x_0, x_i)$ is an $SE(N)$-invariant function if and only if there exists $k_i(\cdot)$, $i \in \{1, \ldots p\}$, such that for all $x_0, x_1, \ldots, x_p \in \mathbb{R}^N$ we have*

$$h_i(x_0, x_i) = h_i(x_0, x_0) + k_i(\|x_i - x_0\|)(x_i - x_0) \quad (26)$$

for all $i \in \{1, \ldots, p\}$ and

$$\sum_{i=1}^p h_i(x_0, x_0) = 0\,. \quad (27)$$

Proof. As before, the quasi-linearity of S, which follows from Eq. (26) and (27), trivially implies its $SE(N)$-invariance. We will prove the converse by induction with respect to p. The base step $p = 2$ is established by Lemma V.2. For the induction step, we assume that Lemma V.3 holds for p and we must show that it also holds for $p + 1$.

Let $x_{p+1} = x_1$ and define the function $h'_1(x_0, x_1) = h_1(x_0, x_1) + h_{p+1}(x_0, x_1)$. Clearly $h'_1(x_0, x_1) + \sum_{i=2}^p h_i(x_0, x_i)$ is an $SE(N)$-invariant function and by the induction hypothesis we have for all $i \in \{2, \ldots, p\}$

$$\begin{aligned} h_i(x_0, x_i) &= h_i(x_0, x_0) + k_i(\|x_i - x_0\|)(x_i - x_0) \\ h'_1(x_0, x_1) &= h'_1(x_0, x_0) + k'_1(\|x_1 - x_0\|)(x_1 - x_0) \\ &= h_1(x_0, x_0) + h_{p+1}(x_0, x_0) \\ &\quad + k'_1(\|x_1 - x_0\|)(x_1 - x_0) \end{aligned}$$

and $h'_1(x_0, x_0) + \sum_{i=2}^p h_i(x_0, x_0) = \sum_{i=1}^{p+1} h_i(x_0, x_0) = 0$.

Similarly, let $x_{p+1} = x_2$ and define $h'_2(x_0, x_2) = h_2(x_0, x_2) + h_{p+1}(x_0, x_2)$. Using the same argument as before, we obtain $h_1(x_0, x_1) = h_1(x_0, x_0) + k_1(\|x_1 - x_0\|)(x_1 - x_0)$. Substituting h_1 in the expression of h'_1 and solving for h_{p+1} we have

$$\begin{aligned} h_{p+1}(x_0, x_{p+1}) &= h'_1(x_0, x_{p+1}) - h_1(x_0, x_{p+1}) \\ &= h_{p+1}(x_0, x_0) \\ &\quad + k_{p+1}(\|x_{p+1} - x_0\|)(x_{p+1} - x_0), \end{aligned}$$

where $k_{p+1} = k'_1 - k_1$. This concludes the proof. □

We conclude this section with a characterization theorem of the total interaction functions of pairwise interaction systems.

Theorem V.4. *Let $S(x_0, x_1, \ldots, x_p) = \sum_{j=1}^p h_j(x_0, x_j)$, where $h_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $p \geq 1$. Then S is $SE(N)$-invariant if and only if it is the sum of quasi-linear functions in $x_j - x_0$, $j \in \{1, \ldots, p\}$, that is $S = \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0)$, where $k_j : \mathbb{R}_{\geq 0} \to \mathbb{R}$.*

Proof. Let $S(x_0, \ldots, x_p) = \sum_{j=1}^p h_j(x_0, x_j)$ be an $SE(N)$-invariant function, it follows from Lemma V.3 that there exists $k_j(\cdot)$ for all $j \in \{1, \ldots, p\}$, such that

$$\begin{aligned} S &= \sum_{j=1}^p \left(h_j(x_0, x_0) + k_j(\|x_j - x_0\|)(x_j - x_0)\right) \\ &= \sum_{j=1}^p h_j(x_0, x_0) + \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0) \\ &= \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0), \end{aligned}$$

where the last equality follows from Eq. (27) of Lemma V.3, which says that the sum of all affine terms must vanish.

Conversely, let $S = \sum_{j=1}^p k_j(\|x_j - x_0\|)(x_j - x_0)$, then S is $SE(N)$-invariant, i.e. for all $(R, w) \in SE(N)$

$$\begin{aligned} RS &= \sum_{j=1}^p k_j(\|x_j - x_0\|)\; R\,(x_j - x_0) \\ &= \sum_{j=1}^p k_j(\|Rx_j + w - (Rx_0 + w)\|)(Rx_j + w - (Rx_0 + w)) \\ &= S(Rx_0 + w, Rx_1 + w, \ldots, Rx_p + w), \end{aligned}$$

where we used the fact that $\|Rx\| = \|x\|$ for all $R \in SO(N)$ and $x \in \mathbb{R}^N$. The proof is now complete. □

Thm. III.7 follows immediately from Thm. V.4, since we can apply Thm. V.4 on the total interaction function S_i of any agents i, where p, x_0 and $h_j(x_0, x_j)$, $j \in \{1, \ldots, p\}$, correspond to $|\mathcal{N}_i^{\rightarrow}|$, x_i and $f_{ij}(x_i, x_j)$, $j \in \mathcal{N}_i^{\rightarrow}$, respectively.

Remark V.5. *Theorem III.7 is stated in terms of total interaction functions, independent of a notion of dynamics, which has two benefits: (1) it greatly expands the applicability of the result to other cases (See. VII), and (2) we do not need to assume any smoothness conditions on the functions, such as continuity or differentiability.*

VI. Stability of $SE(N)$-invariant systems

In this section, we explore the stability of $SE(N)$-invariant pairwise interaction systems, showing that a subclass of such systems converges to a consensus state (one in which all agents' states are equal). The stability result exploits the structure of $SE(N)$-invariant systems imposed by Thm. III.7 and some additional constraints on the connectivity of the communication graph and local interaction functions.

Before we state the stability theorem, we prove a lemma connecting the Laplacian matrix of a graph with the convergence rate of the systems towards the equilibria set.

Lemma VI.1. *Let $\mathcal{L}$ be a $n \times n$ real symmetric positive semidefinite matrix with eigenvalues $\lambda_n \geq \ldots \geq \lambda_2 > \lambda_1 = 0$ and $\mathbf{1}_n$ be the right eigenvector corresponding to the eigenvalue $\lambda_1 = 0$. Then for all $x \in \mathbb{R}^{N\times n}$, $N > 2$, such that $(\mathbf{1}_n^T \otimes I_N)x = 0$, we have*

$$x^T(\mathcal{L} \otimes I_N)x \geq \lambda_2(\mathcal{L})\left\|x\right\|^2. \tag{28}$$

Proof. The spectrum of the Kronecker product of two matrices A, B is composed of the pairwise product of eigenvalues of A and B. Therefore, $\mathcal{L} \otimes I_N$ has the same eigenvalues as $\mathcal{L}$. The inequality in Eq. (28) follows from a special case of the Courant-Fisher theorem [8], [21].

□

Theorem VI.2. *Let (G, F) be a continuous-time pairwise-interaction system that satisfies the following properties:*

1) *(G, F) is $SE(N)$-invariant;*
2) *G is strongly connected;*
3) *(G, F) is balanced, i.e. for all agents i and $x_i, x_j \in \mathbb{R}^N$*

$$\sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) + \sum_{j\in\mathcal{N}_i^{\leftarrow}} f_{ji}(x_j, x_i) = 0 \tag{29}$$

4) *positivity – for all $(i, j) \in E(G)$ and $x_i \neq x_j \in \mathbb{R}^N$*

$$(x_j - x_i)^T f_{ij}(x_i, x_j) > 0. \tag{30}$$

Then the consensus set $\Omega(\bar{x}(0)) = \{x | x_i = \bar{x}(0), \forall i \in V(G)\}$ is globally asymptotically stable, where $x = [x_1^T, \ldots, x_n^T]^T$ is the stacked state vector and $\bar{x}(0) = \frac{1}{n}\sum_{i\in V(G)} x_i(0)$, $n = |V(G)|$. Moreover, for each $(i, j) \in E(G)$ the limit $\sigma_{ij} = \lim_{x_i\to x_j} \frac{(x_j-x_i)^T(f_{ij}(x_i,x_j)-f_{ij}(x_i,x_i))}{\|x_j-x_i\|^2}$ exists, and if $\sigma_{ij} > 0$ for all $(i, j) \in E(G)$, then $\Omega(\bar{x}(0))$ is globally exponentially stable.

Proof. The proof uses a Lyapunov function based argument similar to the one in [8, Thm. 3]. We use Thm. III.7 to rewrite the dynamics of the system in quasi-linear form. We proceed to define a weighted Laplacian matrix, where the weights are dependent on the agents' states, which is the main difference from the proof presented in [8]. Finally, we define a quadratic Lyapunov function and show that the total derivative can be upper bounded using the Fiedler value of the Laplacian matrix and thus guarantees global asymptotic stability. The details are presented below.

First, we show that the average state $\bar{x}(t) = \frac{1}{n}\sum_{i\in V(G)} x_i(t)$ is invariant with respect to time. The derivative of $\bar{x}(t)$ is

$$\begin{aligned}\dot{\bar{x}} &= \frac{1}{n}\sum_{i\in V(G)}\sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) = \frac{1}{n}\sum_{(i,j)\in E(G)} f_{ij}(x_i, x_j)\\ &= \frac{1}{2n}\sum_{i\in V(G)}\left(\sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) + \sum_{j\in\mathcal{N}_i^{\leftarrow}} f_{ji}(x_j, x_i)\right)\\ &= 0,\end{aligned}$$

where the third equality follows from writing the sum of all local interaction functions in two ways, using the incoming and outgoing edges. The last equality follows from the assumption that (G, F) is balanced.

Let $\delta(t) = x(t) - \mathbf{1}_n \otimes \bar{x}(0)$ be the *disagreement* vector.The next step is to show that the *disagreement space* spanned by δ is orthogonal to the consensus space

$$\begin{aligned}(\mathbf{1}_n^T \otimes D_\alpha)\delta(t) &= (\mathbf{1}_n^T \otimes D_\alpha)x(t) - (\mathbf{1}_n^T \otimes D_\alpha)(\mathbf{1}_n \otimes \bar{x}(0))\\ &= D_\alpha\left(n\bar{x}(t) - n\bar{x}(0)\right) = 0,\end{aligned}$$

where $\alpha \in \mathbb{R}^n$ and $D_\alpha = \mathrm{diag}(\alpha)$. The last equality above holds due to the conservation of the average state.

Next, we use the $SE(N)$-property to rewrite the system's dynamics in the quasi-linear form given by Thm. V.4. Let $L(x)$ denote the $n \times n$ weighted Laplacian matrix of (G, F), i.e. for all $i, j \in V(G)$

$$L_{ij} = \begin{cases}\sum_{p\in\mathcal{N}_i^{\rightarrow}} k_{ip}(\|x_i - x_p\|) & \text{for } i = j\\ -k_{ij}(\|x_i - x_j\|) & \text{for } i \neq j \text{ and } (i, j) \in E(G)\\ 0 & \text{otherwise}\end{cases}$$

where $\sum_{j\in\mathcal{N}_i^{\rightarrow}} f_{ij}(x_i, x_j) = \sum_{j\in\mathcal{N}_i^{\rightarrow}} k_{ij}(\|x_j - x_i\|)(x_j - x_i)$. The positivity assumption in Eq. (30) implies that $k_{ij}(a) > 0$ for all $(i, j) \in E(G)$ and $a > 0$.

Using the Laplacian, the system dynamics may be written in the following compact form:

$$\dot{x} = -(L(x) \otimes I_N)x. \tag{31}$$

Also, because $k_{ij}(\|x_i - x_j\|) = k_{ij}(\|x_i + \alpha - (x_j + \alpha)\|)$, we have that $L(x) = L(x + (\mathbf{1}_n \otimes \alpha))$, for all $\alpha \in \mathbb{R}^N$. This implies that $L(x) = L(\delta)$. Moreover, the dynamics of the *disagreement* vector is

$$\dot{\delta} = \dot{x} = -(L(x) \otimes I_N)x \tag{32}$$
$$= -(L(\delta) \otimes I_N)\big(\delta + (\mathbf{1}_n \otimes \bar{x}(0))\big) \tag{33}$$
$$= -(L(\delta) \otimes I_N)\delta + (L(\delta) \otimes I_N)(\mathbf{1}_n \otimes \bar{x}(0)) \tag{34}$$
$$= -(L(\delta) \otimes I_N)\delta, \tag{35}$$

where the second term in Eq. (34) vanishes, because $\mathbf{1}_n$ is a right eigenvector of $L(\delta)$.

Let $\hat{L}(x) = \frac{1}{2}(L(x) + L^T(x))$ be the Laplacian matrix of the mirror graph of G, i.e. the graph with both the edges of G and the reversed edges of G. Notice that $\hat{L}(x)$ is symmetric and $x^T\hat{L}(x)x = \frac{1}{2}(x^T L(x)x + x^T L^T(x)x) = x^T L(x)x$.

Consider the Lyapunov function $V(\delta) = \frac{1}{2}\|\delta\|^2$, which is trivially positive definite and radially unbounded. The total derivative of $V(\cdot)$ is

$$\dot{V}(\delta) = \delta^T\dot{\delta} = -\delta^T(L(\delta)\otimes I_N)\delta \tag{36}$$
$$= -\delta^T(\hat{L}(\delta)\otimes I_N)\delta \tag{37}$$
$$\leq -\lambda_2(\hat{L}(\delta))\|\delta\|^2, \tag{38}$$

where $\lambda_2(\hat{L}(\delta))$ is the Fiedler value (second smallest eigenvalue) of $\hat{L}(\delta)$. The inequality in Eq. (38) follows from Lemma VI.1, because G is balanced and thus $\hat{L}(x)\mathbf{1}_n = \frac{1}{2}(L(x)\mathbf{1}_n + L(x)^T\mathbf{1}_n) = 0$.

The total derivative of the Lyapunov function $\dot{V}(\delta)$ is zero if and only if either: (1) δ is zero, or (2) G is not strongly connected. However, the positivity condition, Eq. (30), implies that G is strongly connected for all $\delta \neq 0$. Since, $\delta = 0$ implies $\dot{\delta} = 0$ it follows from LaSalle's invariance principle that $\delta^* = 0$ is globally asymptotically stable. It follows that $x^* = \mathbf{1}_n\otimes\bar{x}(0)$ and $\Omega(\bar{x}(0))$ is globally asymptotically stable.

Lastly, the limits σ_{ij} exists for all $(i,j)\in E(G)$, because

$$\sigma_{ij} = \lim_{x_i\to x_j}\frac{(x_j - x_i)^T\left(f_{ij}(x_i,x_j) - f_{ij}(x_i,x_i)\right)}{\|x_j - x_i\|^2} \tag{39}$$
$$= \lim_{x_i\to x_j} k_{ij}(\|x_j - x_i\|)\frac{(x_j - x_i)^T(x_j - x_i)}{\|x_j - x_i\|^2} \tag{40}$$
$$= k_{ij}(0). \tag{41}$$

Notice that the set $\Lambda_2 = \{\lambda_2(L(\delta(t)))|t\geq 0\}$ is compact and therefore admits a minimum value. If $\sigma_{ij} = k_{ij}(0) > 0$, then all values in Λ_2 are positive. In particular, it follows that $\min\Lambda_2 > 0$. We can then upper bound the quantity in Eq. (38) by $\dot{V}\leq -\min\Lambda_2\|\delta\|^2$, which in turn shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\| \leq -\min\Lambda_2\|\delta\|. \tag{42}$$

Therefore, $\Omega(\bar{x}(0))$ is globally exponentially stable. □

VII. Extensions

The main result presented in Sec. III is stated for first order (kinematic) continuous-time dynamics. In this section, we discuss extensions to discrete-time and higher order dynamics. We also show that the results hold for switching and time-varying graph topologies.

A. Discrete-time systems

A discrete-time pairwise interaction system can be defined by replacing differentiation ($\dot{x}_i$) with one-step difference ($\Delta x_i(t) = x_i(t+1) - x_i(t)$) in Eq. (2) of Def. III.2. The definitions of the total interaction function and $SE(N)$-invariance remain unchanged, (see Eq. (3) of Def. III.2 and Def. III.3, respectively).

The main result of the paper, Thm. III.7, holds for discrete-time systems as well. The stability results on the other hand need to be adjusted.

Lemma VII.1. *Let (X,d) be a non-empty complete metric space and $(T_n)_{n\geq 0}$ be a sequence of Lipschitz continuous functions such that all admit a Lipschitz constants $q < 1$. Define the sequence $x_{n+1} = T_n(x_n)$. If all maps T_n have the same fixpoint $x^*\in X$, then for all $x_0\in X$ we have $x_n\to x^*$.*

Proof. First note that all maps T_n are contractions, because $q < 1$. Thus, by the contraction mapping theorem, all T_n have a unique fixpoint x^*. Moreover, for all $n\geq 0$ and $x\in X$ we have the following

$$d(T_n(x),x^*) = d(T_n(x),T(x^*)) \leq qd(x,x^*). \tag{43}$$

It follows by induction $d(x_n,x^*)\leq q^n d(x_0,x^*)$, for all $n\geq 1$. The base case $n=1$ follows by applying Eq. (43). For the induction step, we again use Eq. (43), $d(x_{n+1},x^*) = d(T_n(x_n),x^*)\leq qd(x_n,x^*)\leq q^{n+1}d(x_0,x^*)$, where in the last inequality we used the induction hypothesis.

Lastly, x_n is a Cauchy sequence, because for all $m,n\geq 0$ $d(x_m,x_n)\leq d(x_m,x^*) + d(x^*,x_n)\leq (q^m + q^n)d(x_0,x^*)$, where we used the triangle inequality in the first inequality. Therefore, x_n has the unique limit x^*, because X is complete and the distance map d is continuous. □

Definition VII.2. *Let (G,F) be a discrete-time pairwise interaction system and G^T be the transpose graph of G, i.e. the graph with all edges reversed. Denote by S^G and S^{G^T} the vectors of stacked total interaction functions for all agents with communication graphs G and G^T, respectively. System (G,F) is said to be* forward-backward consistent *if*

$$\left(\mathbf{id} + S^{G^T}\right)\circ\left(\mathbf{id} + S^G\right) = \left(\mathbf{id} + S^G\right)\circ\left(\mathbf{id} + S^{G^T}\right), \tag{44}$$

where $\mathbf{id}$ *is the identity function and $\circ$ is the function composition operator.*

Remark VII.3. *The identity function in the terms of Eq.* (44) *arises, because the equations of the forward (G) and backward (reversed, G^T) evolution of the system are $x(t+1) = x(t) + S^G(x(t))$ and $x(t+1) = x(t) + S^{G^T}(x(t))$, respectively.*

Remarks VII.4. *Def. VII.2 describes a property about the evolution of a system in two time units, where in either the first or the second time unit the edges of the communication graph are reversed. The property in Eq.* (44) *captures the idea that the state the system ends up in is independent of when the reversal of the edges occurred.*

The property can also interpreted in the following way. Consider a network with half-duplex communication links and a global switch which changes the direction of all links at the same time. The forward-backward consistency property implies that the state of the network at time t depends only on the initial state and the number of network switches until time t and not the sequence of switches itself.

Yet another way to interpret the property is as a relaxation of time-reversibility. If the two terms in Eq. (44) *were equal to the identity function, then the pairwise interaction system (G,F) would be time-reversible and moreover the system could be brought back to the initial state using (G^T,F) with the communication graph reversed. Therefore, Def. VII.2 can be though of as a relaxation of time-reversibility.*

Theorem VII.5. *Let (G,F) be a discrete-time pairwise interaction system that satisfies the following properties:*

1) *(G,F) is $SE(N)$-invariant;*

2) *G is strongly connected;*
3) *(G, F) is forward-backward consistent, see Def. VII.2;*
4) *positivity – for all $(i, j) \in E(G)$*

$$\inf_{x_i \neq x_j} \left\{ \frac{(x_j - x_i)^T (f_{ij}(x_i, x_j) - f_{ij}(x_i, x_i))}{\|x_j - x_i\|^2} \right\} > 0 \tag{45}$$

5) *the* maximum out-degree *is less than one, i.e.*

$$\sup_{i,\, x_i} \left\{ \sum_{j \in \mathcal{N}_i^{\rightarrow}} \frac{\|f_{ij}(x_i, x_j) - f_{ij}(x_i, x_i)\|}{\|x_j - x_i\|} \right\} < 1. \tag{46}$$

Then the consensus set $\Omega(\bar{x}(0)) = \{x | x_i = \bar{x}(0), \forall i \in V(G)\}$ is globally exponentially stable, where $x = [x_1^T, \ldots, x_n^T]^T$ is the stacked state vector and $\bar{x}(0) = \frac{1}{n}\sum_{i \in V(G)} x_i(0)$, $n = |V(G)|$.

Proof. In the following we use the notation introduced in the proof of Thm VI.2. Thus, the dynamics can be written as

$$x(t+1) = (P(x(t)) \otimes I_N) x(t) \tag{47}$$

$$\delta(t+1) = (P(\delta(t)) \otimes I_N) \delta(t), \tag{48}$$

where $P(x) = I_n - L$ is the Perron matrix. Similarly to $L(\cdot)$, we have $P(x) = P(\delta)$.

For any fixed $\delta \in \mathbb{R}^{n \times N}$ such that $(\mathbf{1}_n^T \otimes I_N)\delta = 0$, we have that $P(\delta)$ is a nonnegative doubly stochastic matrix. The positivity assumption Eq. (45) is equivalent to $k_{ij}(a) > 0$ for all $a \geq 0$ and $(i, j) \in E(G)$, which trivially implies that all off-diagonal elements of $P(\delta)$ are non negative. Moreover, the maximum degree assumption can be restated as $\sum_{j \in N_i} k_{ij}(\|\delta_i - \delta_j\|) < 1$ which is equivalent to $P_{ii}(\delta) > 0$. The forward-backward consistency property implies that $P(\delta)$ is a normal matrix, for all δ. The Perron matrix $P(\delta)$ is double stochastic, i.e. G is balanced, because $\mathbf{1}_n$ is a right eigenvector of $L(\delta)$ and $P(\delta)P^T(\delta)\mathbf{1}_n = P^T(\delta)P(\delta)\mathbf{1}_n = P^T(\delta)\mathbf{1}_n$ which implies that $P^T(\delta)\mathbf{1}_n = a\mathbf{1}_n$, $a \neq 0$, is an eigenvector of $P(\delta)$ corresponding to the eigenvalue 1. Since $P^T(\delta)$ has the same spectrum as $P(\delta)$, it follows that a must be 1.

The Perron matrix is a contraction on the linear space defined by $(\mathbf{1}_n^T \otimes I_N)\delta = 0$, because

$$\|(P(\delta) \otimes I_N)\alpha\|^2 = \alpha^T (P(\delta) \otimes I_N)^T (P(\delta) \otimes I_N)\alpha \tag{49}$$

$$= \alpha^T ((P(\delta)^T P(\delta)) \otimes I_N)\alpha \tag{50}$$

$$= \alpha^T ((U D^* D U^*) \otimes I_N)\alpha \tag{51}$$

$$\leq |\mu_2(P(\delta))|^2 \cdot \|\alpha\|^2, \tag{52}$$

where $\mu_2(P(\delta))$ is the second largest eigenvalue in absolute value of $P(\delta)$, $P(\delta) = UDU^*$, U is a unitary matrix, D is the diagonal matrix corresponding to the spectrum of $P(\delta)$, and * is the conjugate transpose operator. The inequality in Eq. (52) follows from the Courant-Fisher Theorem [21].

Lastly, it follows that $P(\delta(t)) \otimes I_N$ is a sequence of contraction maps. All of them admit 0 as a fixpoint. The Lipschitz constant for all of them is $q = \sup_{t \geq 0} |\mu_2(P(\delta(t)))| < 1$, because the positivity assumption guarantees that $P(\delta(t))$ is nonnegative doubly stochastic for all $t \geq 0$. By Lemma VII.1 it follows that $\delta(t)$ converges to 0, where $X \subset \mathbb{R}^N$ is the space defined by $(I_N \otimes D_\alpha)\delta = 0$ with distance function induced by the Euclidean norm $\|\cdot\|$. □

B. Higher-order dynamics

In this section we extend the notion of $SE(N)$-invariance to higher-order pairwise interaction systems, i.e. each agent's dynamics has order $m \geq 2$. If the dynamics of these systems depends only the agents' states, then the definitions and results from Sec. III all hold. However, we are interested in systems whose dynamics depend on the agents' (generalized) velocities as well. For this class of systems, we show a similar result to Thm. III.7. As in Sec. V, all (generalized) velocities are measured with respect to a global inertial frame, but are represented in a reference frame of the agents' choice.

Definition VII.6 ($SE(N)$-invariant function)**.** *A function $f : \mathbb{R}^{N \times m \times p} \to \mathbb{R}^N$ is said to be $SE(N)$-invariant if for all $R \in SO(N)$ and all $w \in \mathbb{R}^N$ the following condition holds:*

$$Rf(x, v^1, \ldots, v^{m-1}) = f(\mathbf{R}x + \mathbf{w}, \mathbf{R}v^1, \ldots, \mathbf{R}v^{m-1}), \tag{53}$$

where $x, v^1, \ldots, v^{m-1} \in \mathbb{R}^N$, $\mathbf{R} = R \otimes I_p$ and $\mathbf{w} = w \otimes \mathbf{1}_p$.

Definition VII.7 (Pairwise Interaction System)**.** *A continuous-time pairwise interaction system is a double (G, F), where G is a graph and $F = \{(f_{ij}^0, \ldots, f_{ij}^{m-1}) \mid f_{ij}^r : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, (i, j) \in E(G)\}$ is a set of tuple of functions associated to its edges. Each $i \in V(G)$ labels an agent, and a directed edge (i, j) indicates that node i requests and receives information from node j. The dynamics of each agent are described by*

$$x_i^{(m)} = \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}^0(x_i, x_j) + \sum_{r=1}^{m-1} \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}^r(x_i^{(r)}, x_j^{(r)}), \tag{54}$$

where f_{ij}^r, $0 \leq r < m$, define the influence (interaction) of j on i.

For each agent $i \in V(G)$, we denote the *total interaction* on agent i by

$$S_i(x, v^1, \ldots, v^{m-1}) = \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}^0(x_i, x_j) + \sum_{r=1}^{m-1} \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}^r(v_i^r, v_j^r).$$

The definitions of $SE(N)$-invariant systems and quasi-linear systems remain unchanged, but are interpreted using the extended notions. The main theorem can thus be extended as follows:

Theorem VII.8. *Let (G, F) be a continuous-time pairwise interaction system such that $f_{ij}^r(v_i^r, v_j^r) = g_{ij}^r(v_i^r - v_j^r)$, where $g_{ij}^r : \mathbb{R}^N \to \mathbb{R}^N$ and $r \in \{1, \ldots, m-1\}$. Then (G, F) is $SE(N)$-invariant if and only if it is quasi-linear.*

Proof. Let S_i be the total interaction function of agent $i \in V(G)$. Let $v^r = 0$ for all $1 \leq r \leq m-1$. Since

$$RS_i(x, 0, \ldots, 0) = S_i(\mathbf{R}x + \mathbf{w}, 0, \ldots, 0)$$

for all $(R, w) \in SE(N)$, we have by Lemma V.3 that

$$S_i(x, 0, \ldots, 0) = \sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}^0(\|x_i - x_j\|)(x_j - x_i).$$

Similarly, let $x = 0$ and $v^r = 0$ for $r \neq s$, $1 \leq r, s \leq m-1$. We have

$$\begin{aligned} RS_i(0, 0, \ldots, v^s, \ldots, 0) &= R \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}^s(v_i^s, v_j^s) \\ &= \sum_{j \in \mathcal{N}_i^{\rightarrow}} f_{ij}^s(Rv_i^s, Rv_j^s) \\ &= \sum_{j \in \mathcal{N}_i^{\rightarrow}} g_{ij}^s(Rv_i^s + w - (Rv_j^s + w)) \\ &= S_i(0, \ldots, \mathbf{R}v^s + \mathbf{w}, \ldots, 0) \end{aligned}$$

for all $(R, w) \in SE(N)$. Again, by Lemma V.3 it follows that

$$S_i(0, 0, \ldots, v^s, \ldots, 0) = \sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}^s(\|v_j^s - v_i^s\|)(v_j^s - v_i^s).$$

Overall, it follows that

$$\begin{aligned} S_i = &\sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}^0(\|x_i - x_j\|)(x_j - x_i) \\ &+ \sum_{r=1}^{m-1} \sum_{j \in \mathcal{N}_i^{\rightarrow}} k_{ij}^r(\|v_i^r - v_j^r\|)(v_j^r - v_i^r). \end{aligned}$$

Conversely, if all total interaction functions are quasi-linear, it is straightforward to check that the system is $SE(N)$-invariant. □

C. Switching topologies

The main result of the paper, Thm. III.7, as well as the extensions to discrete-time systems and higher order system will hold also in the case when the communication topology G changes and the switching signal is time-dependent. Intuitively, the time-varying topology is not related to the reference frames of the agents. Thus, $SE(N)$-invariance implies the quasi-linear structure regardless of the topology of the system.

VIII. Examples

In this section we provide some examples to clarify and illustrate the notions of $SE(N)$-invariance and quasi-linearity for pairwise interaction systems. We also consider several existing pairwise multi-agent systems that have been studied in the literature. We show that many of these are $SE(N)$-invariant, although we also show an example that is not, and one that is only $SE(N)$-invariant under certain conditions. These results are summarized in Table I.

The following example shows an $SE(N)$-invariant system with local interaction functions which are not quasi-linear. However, as shown by Thm. V.4, the total interaction functions associated with the system's agents can be rewritten as sums of quasi-linear functions. Moreover, Ex. VIII.1 provides an example of a weakly stable system where the agents follow elliptical periodic orbits (see Fig. 4). The shape of the elliptical orbits depends on the initial states of the agents: (1) if the agents start from equidistant states then they follow circular periodic trajectories (see Fig. 4(a)); (2) otherwise their periodic trajectories are elliptical (see Fig. 4(b)). This example, together with the systems considered in [12] and [13], show that $SE(N)$-invariant pairwise interaction systems have rich asymptotic behaviors aside from consensus.

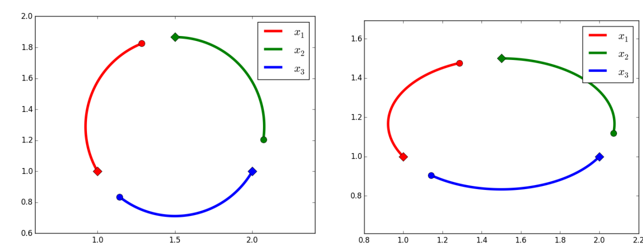
Example VIII.1. *Let (G, F) be a pairwise interaction system where $G = K_3$ is the complete graph with 3 vertices and*

$$f_{ij}(x_i, x_j) = \begin{cases} x_j & (i,j) \in \{(1,2), (2,3), (3,1)\} \\ -x_j & \text{otherwise} \end{cases}$$

The pairwise interaction functions of this system are not quasi-linear in $x_j - x_i$, $(i, j) \in E(G)$. However, the system can easily be checked to be $SE(N)$-invariant. For agent 1 we have

$$\begin{aligned} S_1(x_1, x_2, x_3) &= f_{12}(x_1, x_2) + f_{13}(x_1, x_3) = x_2 - x_3 \\ RS_1 &= Rx_2 + w - (Rx_3 + w) \\ &= f_{12}(Rx_1 + w, Rx_2 + w) + f_{13}(Rx_1 + w, Rx_3 + w) \\ &= S_1(Rx_1 + w, Rx_2 + w, Rx_3 + w), \end{aligned}$$

where $R \in SO(N)$ and $w \in \mathbb{R}^N$. However, by Thm. V.4 the total interaction function S_1 must be a sum of quasi-linear functions. Indeed, we can rewrite $S_1 = x_2 - x_1 + (-1)(x_3 - x_1)$. Similarly, the $SE(N)$-property holds for the total interaction functions of the other two agents and these functions can be rewritten as sums of quasi-linear functions.



(a) $x_1^T(0) = [1, 1]$, $x_2^T(0) = [\frac{3}{2}, 1 + \frac{\sqrt{3}}{2}]$, $x_3^T(0) = [2, 1]$ (b) $x_1^T(0) = [1, 1]$, $x_2^T(0) = [\frac{3}{2}, \frac{3}{2}]$, $x_3^T(0) = [2, 1]$

Fig. 4. Trajectories of the $SE(2)$-invariant system presented in Ex. VIII.1. The three agents are shown in red, blue and green, respectively. The states of the agents at time $t = 0$ sec and $t = 1$ sec are marked by diamonds and dots, respectively.

Example 1 in Tab. I was proposed in [12] to model swarm aggregation and is a quasi-linear system because $g(\cdot)$ is a quasi-linear function. The system exhibits an asymptotic behavior where the agents aggregate (in finite time) within a hyper-ball and stay inside it forever [12]. The second [15], third [10] and fourth [11] examples define the agents' dynamics based on potential functions. Example 2 from [15] drives the agents towards some goal states which are encoded in the $\gamma_i()$ functions, while ensuring that the agents avoid each other and fixed and known obstacles and it is enforced using the $\beta_i()$ functions. The system is not quasi-linear, because the potential function whose gradient is used for navigation depends explicitly on the agents' states, as opposed to distances between agents' states, and thus its gradient cannot be a quasi-linear function. We can conclude that the multi-agent system in the second example is not $SE(N)$-invariant. On the other hand, example 4 [11] is quasi-linear, because the gradients of $\nabla_{x_i} V_{ij}(\cdot)$ are quasi-linear functions.

TABLE I

THE TABLE CONTAINS EXAMPLES OF NETWORKED SYSTEMS THAT ARE QUASI-LINEAR, EXCEPT FOR THE SECOND EXAMPLE AND POSSIBLY THE FOURTH. IT FOLLOWS THAT THE QUASI-LINEAR SYSTEMS BELOW ARE ALSO $SE(N)$-INVARIANT BY THM. III.7. ALL SYSTEMS HAVE n AGENTS AND FOR EACH AGENT $i \in \{1, \ldots, n\}$, WE DENOTE BY x_i ITS STATE. THE MAPS $V_I, V_h, V_{ij} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ REPRESENT POTENTIAL FUNCTIONS. IN THE THIRD EXAMPLE, $\widetilde{x}_p$ REPRESENTS THE STATE OF A VIRTUAL LEADER $p \in \{1, \ldots, m\}$. IN THE FOLLOWING, $\nabla_{x_i} V$ REPRESENTS THE GRADIENT OF V WITH RESPECT TO x_i.

No.	System dynamics	Reference	Quasi-linear?
1	$\dot{x}_i = \sum_{j=1}^{n} g(x_i - x_j)$ $g(y) = -y\left(a - b\exp\left(-\frac{\\|y\\|^2}{c}\right)\right)$	[12]	Yes
2	$\dot{x}_i = -\alpha \nabla_{x_i}\left(\frac{\gamma_i(x)}{(\gamma_i(x)^k + \beta_i(x))^{1/k}}\right)$	[15]	No
3	$\ddot{x}_i = -\sum_{j \neq i}^{n} \nabla_{x_i} V_I(\\|x_i - x_j\\|) - \sum_{k=0}^{m-1} \nabla_{x_i} V_h(\\|x_i - \widetilde{x}_k\\|)$ $\ddot{\widetilde{x}}_p = \widetilde{f}_p(x_j, \widetilde{x}_k, \dot{x}_j, \dot{\widetilde{x}}_k)$ where $1 \leq i \leq n$ and $1 \leq p \leq m$	[10]	Yes or No.
4	$\ddot{x}_i = -\sum_{j \in \mathcal{N}_i^{\to}} \nabla_{x_i} V_{ij}(\\|x_i - x_j\\|) - \sum_{j \in \mathcal{N}_i^{\to}} (\dot{x}_i - \dot{x}_j)$	[11]	Yes.
5	$\dot{x}_i = u_i$ or $x_i(k+1) = x_i(k) + u_i$ $u_i = \sum_{j \in \mathcal{N}_i^{\to}} a_{ij}(x_i - x_j)$ or $u_i = \sum_{j \in \mathcal{N}_i^{\to}} (\\|x_i - x_j\\|^2 - d_{ij})(x_i - x_j)$ where $a_{ij} \in \mathbb{R}$ and $d_{ij} \in \mathbb{R}$	[8], [9], [13]	Yes
6	$\ddot{x}_i = \frac{1}{m_i} \sum_{j=1, j \neq i}^{n} \frac{G m_i m_j}{\\|x_i - x_j\\|^3}(x_j - x_i)$	[7]	Yes

We conclude that the system is $SE(N)$-invariant in the sense of Def. VII.7 by Th. VII.8 for higher order systems with generalized velocities. The system in example 3 is quasi-linear if and only if the dynamics of the virtual leaders $\widetilde{f}_p$ are sums of quasi-linear functions, $1 \leq p \leq m$. Example 5 corresponds to systems implementing consensus and formation control [8], [9], [13]. It is easy to see that these systems are quasi-linear and therefore $SE(N)$-invariant. The last example shows a system of n point masses which interact with each other due to gravity. This system is also quasi-linear and thus exhibits $SE(N)$-invariance, a fact which is well known in Hamiltonian mechanics [7].

IX. CONCLUSIONS

In this paper, we studied the $SE(N)$-invariance property of multi-agent, locally interacting systems. This property, which guarantees the independence of a system of global reference frames, implies that control laws can be computed and executed locally (i.e., in each agent's frame) using only local information available to the agent. This property is critical in applications in which information about a global reference frame cannot be obtained, e.g., in GPS-denied environments.

The main contribution of the paper is to fully characterize pairwise interaction systems that are $SE(N)$-invariant. We showed that pairwise interaction systems are $SE(N)$-invariant if and only if they have a special *quasi-linear* form. Because of the simplicity of this form form, this result can impact ongoing research on design of local interaction laws. The result can also be used to quickly check if a given networked system is $SE(N)$-invariant. We also described a subset of $SE(N)$-invariant pairwise interaction systems that reach consensus by exploiting their quasi-linear structure. Finally, we extended the results to discrete-time and high-order systems and systems with time-dependent switching topologies. As in the continuous case, we proved the convergence to consensus for a subclass of discrete-time $SE(N)$-invariant pairwise interaction systems.

X. APPENDIX. THE CASE $N = 2$

In this section we treat the case $N = 2$. The difference between the cases $N = 2$ and $N \geq 3$ arises from the fact that $SO(2)$, the group of planar rotations, is Abelian, while $SO(N)$ for $N \geq 3$ is not, i.e., rotation matrices in 3 or more dimensions do not, in general, commute.

All results in the paper carry over to the case $N = 2$, because $SO(2)$ and its centralizer are Abelian. In all theorems quasi-linear functions are replaced with similar functions from the centralizer of $SO(2)$. In the following we provide the characterization of $C_T(SO(2))$, which supports our claim.

Proposition X.1. *The centralizer of $SO(2)$ with respect to T is the submonoid $\{(k_1(\|x\|)I_2 + k_2(\|x\|)J_2)\, x\}$, where $k_1, k_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.*

Proof. Let $x \in \mathbb{R}^2$, $x \neq 0$, and $u = \frac{x}{\|x\|}$. Then $R_u = \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix}$ is a rotation matrix in $SO(2)$ and $x = R_u \|x\|\, e_1$.

Next, we evaluate $f(x)$

$$
\begin{aligned}
f(x) &= f(R_u \|x\| e_1) = R_u f(\|x\| e_1) \\
&= \begin{bmatrix} u_1 & u_2 \\ -u_2 & u_1 \end{bmatrix} \begin{bmatrix} f_1(\|x\| e_1) \\ f_2(\|x\| e_1) \end{bmatrix} \\
&= \frac{1}{\|x\|} \begin{bmatrix} x_1 f_1(\|x\| e_1) + x_2 f_2(\|x\| e_1) \\ x_2 f_1(\|x\| e_1) - x_1 f_2(\|x\| e_1) \end{bmatrix} \\
&= \frac{1}{\|x\|} \begin{bmatrix} f_1(\|x\| e_1) & f_2(\|x\| e_1) \\ -f_2(\|x\| e_1) & f_1(\|x\| e_1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\triangleq \begin{bmatrix} k_1(\|x\|) & k_2(\|x\|) \\ -k_2(\|x\|) & k_1(\|x\|) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\end{aligned}
$$

where $k_1(\|x\|) \triangleq \frac{f_1(\|x\| e_1)}{\|x\|}$ and $k_2(\|x\|) \triangleq \frac{f_2(\|x\| e_1)}{\|x\|}$. The case $x = 0$ follows from Lemma IV.2. □

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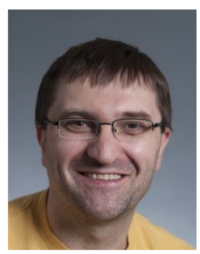

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