Generalized Mean Robustness for Signal Temporal Logic

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***Abstract*— Robustness functions provide quantitative scores to measure the satisfaction of temporal logic formulas. We introduce a general class of parameterized robustness functions for Signal Temporal Logic (STL), and demonstrate how it can be used for control problems involving STL specifications. We employ power means and generalized functional means to capture robust satisfaction over space and time. We show that our general definition encompasses many of the STL robustness functions in the literature. Most importantly, we show how that our notion of robustness addresses the two main limitations of the the traditional robustness (masking and locality), which currently limit using robustness-based approaches for control. The proposed robustness function parameters affect the conservativeness of the score, and can be chosen based on desired performance. We show how the proposed robustness can be used for control.**

***Index Terms*— Signal Temporal Logics, Power mean, Generalized Mean, Robustness, Control Synthesis**

I. Introduction

CYBER-PHYSICAL systems are required to satisfy complex requirements that are beyond stability or reachability. The wide adoption of these systems in many engineering disciplines, from self-driving cars and robotics to biological systems, have led to an increasing need for techniques and tools to study their correct functioning. Formal methods can express a broad spectrum of properties and constraints in cyber-physical systems including time deadlines, sequentiality, and safety [1], [2]. Numerous tools have been developed to provide formal certificates for satisfaction of logical specifications, and to automatically design control policies that guarantee correctness [3]–[5]. Linear Temporal Logic (LTL) [6], Metric Temporal Logic (MTL) [7], Time Window Temporal Logic (TWTL) [8] and Signal Temporal Logic (STL) [9] have been widely used as specification languages due to their expressivity and similarity to natural language.

STL defines temporal and logical properties of real-valued signals. In its qualitative (Boolean) semantics, a signal either satisfies or violates a formula. The STL quantitative semantics [10], known as robustness, provides a measure of satisfaction or violation of a formula. Early works showed that it can be used for monitoring [9]. It has also been utilized as the objective function in different optimization frameworks to synthesize control policies [11]–[14].

The traditional robustness introduced in [10] uses $\max$ and $\min$ functions over temporal and logical formulae, resulting in a *sound* yet non-convex and non-smooth function, which only includes the most critical part of the signal in the robustness. Heuristic optimization algorithms such as Particle Swarm Optimization and Rapidly Exploring Random Trees (RRTs) were initially used to optimize the non-smooth robustness function [15], [16]. Mixed Integer Linear Programming (MILP) encoding of the temporal and Boolean constraints was later proposed in [11], [17]. However, MILP encoding is only applicable to linear systems with linear costs and formulae, and relies on defining integer variables. This method scales poorly with the size and horizon of the specifications. Previous works focused on smoothing the robustness function by using approximations of $\max$ and $\min$, resulting in a differentiable yet not *sound* robustness. This approach enables the use of scalable, gradient-based optimization methods, which are applicable to general nonlinear systems [18], [19].

Averaging was used in [20] to define quantitative semantics for MTL in terms of linear time-invariant filters. Different from STL robustness, the MTL semantics captures the degree of truth, and is non-differentiable. Robustness notions for LTL are proposed in [21], [22] that are closer to *relaxed satisfaction* [23] than STL robustness.

The $\max$ and $\min$ functions in the traditional robustness definition also induce *masking* and *locality* effects. Informally, masking means that only the best or worst operands and time points are selected by the robustness score. Thus, no information about any other operands and time points is visible in the result. The locality of a robustness score implies that it depends only on the value of one predicate at one time point. These have been shown to have negative impacts in optimization problems due to hindering optimizers from obtaining gradient information to improve solutions, and resulting in solutions that were brittle to noise [24], [25]. Some works tackled the aforementioned issues and refined the robustness function to include more information on the signal by employing averaging or cumulative functions [24], [26]–[29]. Other works studied parametric approximations of the traditional robustness that enabled tuning the locality and masking levels [25], [30]. These works focused on defining smooth robustness functions suitable for gradient-based optimizations at the cost of losing monotonicity, and, thus, the interpretation of margin

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of satisfaction and violation, i.e., larger values corresponding to greater satisfaction. Monotonicity also precludes the introduction of local optima due to the score.

In this paper, we propose a generalized robustness function for STL. We employ power means and generalized functional-means to define a unified class of robustness scores parameterized by two continuous values, which 1) removes the locality and masking effects of the traditional robustness score, and 2) enables tuning the desired performance and conservativeness through the design parameters. Conservativeness in this context represents the myopic view induced by masking and locality. It impacts the performance of the system in the presence of noise and disturbance. Traditional robustness captures the satisfaction or violation margin at a singular time point, and is, thus, most conservative. The proposed robustness class based on generalized mean uses all sub-formulas at all relevant time points. Solutions to the corresponding optimal control problems satisfy sub-formulas at more time points and can handle disturbance over the entire specification horizon.

The contributions of the paper are: (1) We propose a unified class of parameterized robustness functions that enables tuning desired conservativeness of the score, and as a result the locality and masking effects. (2) We demonstrate the usefulness of the proposed robustness in open-loop control synthesis case studies based on gradient methods such that the system satisfies the desired STL specification robustly over the entire horizon of the formula.

II. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ be the sets of real, non-negative real, integer, and non-negative integer numbers, respectively. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real function. We define $[f]_+ = \max\{f, 0\}$ and $[f]_- = -[-f]_+$. The vectors of ones and zeros of dimension d, and the $d \times d$ identity matrix are denoted by $\mathbf{1}_d$, $\mathbf{0}_d$, $\mathbb{I}_d$, respectively. Let $\delta_\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the Dirac delta function such that $\delta_\tau(t) = 1$ if $t = \tau$, and zero otherwise. Vectors and vector-valued functions are denoted in bold, e.g., $\mathbf{x}$ and $\mathbf{s}$. Define $\|\mathbf{x}\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{\frac{1}{p}}$, where $p \in \mathbb{R}$ and $\mathbf{x} = [x_1, \ldots x_d] \in \mathbb{R}^d$. For a function $f : I \to \mathbb{R}$, define $\|f\|_p = \left(\int_I |f|^p(t)\mathrm{d}t\right)^{\frac{1}{p}}$, where $I \subseteq \mathbb{R}$ is an interval, and $p \in \mathbb{R}$. Note that $\|\cdot\|_p$ is a norm iff $p \geq 1$. The length of an interval I is denoted by $|I|$.

Signal Temporal Logic. A *signal* $\mathbf{s} : \mathbb{T} \to \mathbb{M}$ is a function that maps each time point t in the time domain $\mathbb{T}$ to an n-dimensional vector of real values $\mathbf{s}(t) \in \mathbb{M} \subseteq \mathbb{R}^n$. The time domain $\mathbb{T} \subseteq \mathbb{R}_{\geq 0}$ can be continuous or discrete. The set of all signals is denoted by $\mathbb{S}$. Signal temporal logic (STL) [9] is defined and interpreted over signals $\mathbf{s}$. Its syntax is inductively defined as $\varphi := \top \mid \mu \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathcal{U}_I \varphi_2$, where φ, φ_1, φ_2 are STL formulae, $\top$ is the logical *True* value, $\mu := (\ell(\mathbf{s}(t)) \geq 0)$ is a predicate over $\mathbf{s}(t)$, where $\ell : \mathbb{M} \to \mathbb{D}$ is a Lipschitz continuous function. We assume that $\mathbb{D} = [-\alpha, \alpha]$, $\alpha \in \mathbb{R}_{>0}$, such that $a \in \mathbb{D}$ implies $-a \in \mathbb{D}$. $\wedge$ and $\neg$ are the Boolean operators for conjunction and negation. $\varphi_1 \mathcal{U}_I \varphi_2$ is the timed *until* operator with $I = [t_1, t_2]$ a time interval, $t_2 \geq t_1 \geq 0$. The logical *False* can be derived as $\bot = \neg\top$, and the other Boolean operators (e.g., disjunction $\vee$, implication $\Rightarrow$) are defined in the usual way [9]. The time-constrained eventually $\Diamond_I$ and always $\Box_I$ operators are derived as $\Diamond_I \varphi = \top \mathcal{U}_I \varphi$ and $\Box_I \varphi = \neg\Diamond_I \neg\varphi$, respectively. To simplify the notation, time is implicitly understood to belong to the time domain $\mathbb{T}$, i.e., all time intervals I are interpreted as $I \cap \mathbb{T}$. For example, $[t_1, t_2]$ is an ordered sequence $\{t_1, t_1 + \delta t, t_1 + 2\delta t, \ldots, t_2\}$ if $\mathbb{T}$ is discrete and time is uniformly discretized with step δt. The (Boolean) satisfaction of a formula φ by a signal $\mathbf{s}$ starting from time t is denoted by $(\mathbf{s}, t) \vDash \varphi$ (see [9] for a formal definition). Satisfaction and violation at time 0 are denoted by $\mathbf{s} \vDash \varphi$ and $\mathbf{s} \nvDash \varphi$, respectively.

Example 1: Consider a car of negligible size driving in the environment shown in Fig. 1. Assume the car is required to always drive in the assigned lane ($0 \leq y \leq 4$). We can write this requirement as a STL formula $\varphi = \varphi_1 \wedge \varphi_2$, where $\varphi_1 = \Box_{[1,10]} y(t) \geq 0m$ and $\varphi_2 = \Box_{[1,10]} y(t) \leq 4m$. Trajectory s_1 in Fig. 1 satisfies φ_1, i.e., $s_1 \vDash \varphi_1$, but exceeds the lane boundary and violates φ_2, i.e., $s_1 \nvDash \varphi_2$. The same is true for s_2.

The STL quantitative semantics, known as *robustness*, assigns a real value to indicate *how much* a signal satisfies or violates a formula [10].

Definition 1 (STL Robustness): Given a formula φ and a signal $\mathbf{s}$, the robustness score $\rho(\varphi, \mathbf{s}, t)$ at time t is [10]:

$$
\begin{aligned}
&\rho(\top, \mathbf{s}, t) := \rho_\top, \;\; \rho(\mu, \mathbf{s}, t) := \ell(\mathbf{s}(t))), \;\; \rho(\neg\varphi, \mathbf{s}, t) := -\rho(\varphi, \mathbf{s}, t), \\
&\rho(\varphi_1 \wedge \varphi_2, \mathbf{s}, t) := \min(\rho(\varphi_1, \mathbf{s}, t), \rho(\varphi_2, \mathbf{s}, t)), \\
&\rho(\varphi_1 \vee \varphi_2, \mathbf{s}, t) := \max(\rho(\varphi_1, \mathbf{s}, t), \rho(\varphi_2, \mathbf{s}, t)), \\
&\rho(\Box_I \varphi, \mathbf{s}, t) := \inf_{t' \in t+I} \rho(\varphi, \mathbf{s}, t'), \;\; \rho(\Diamond_I \varphi, \mathbf{s}, t) := \sup_{t' \in t+I} \rho(\varphi, \mathbf{s}, t'), \\
&\rho(\varphi_1 \mathcal{U}_I \varphi_2, \mathbf{s}, t) := \sup_{t' \in t+I} \Big(\min\Big\{\rho(\varphi_2, \mathbf{s}, t'), \inf_{t'' \in [t,t']} \rho(\varphi_1, \mathbf{s}, t'')\Big\}\Big)
\end{aligned} \tag{1}
$$

where $\rho_\top = \sup \mathbb{D} = \alpha$ is the maximum robustness.

Theorem 1 (Soundness [10]): The robustness score is sound, i.e., $\rho(\varphi, \mathbf{s}, t) > 0$ implies that signal $\mathbf{s}$ satisfies φ at time t, and $\rho(\varphi, \mathbf{s}, t) < 0$ implies that $\mathbf{s}$ violates φ at time t.

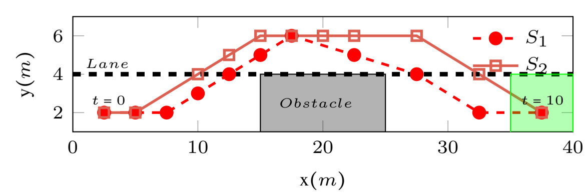

Fig. 1. Sample car trajectories satisfying φ_1 and violating φ_2 in Ex. 1

We refer to $\rho(\varphi, \mathbf{s}, t)$ as *traditional robustness*. For simplicity, $\rho(\varphi, \mathbf{s})$ denotes the robustness at $t = 0$. The formal definition of time *horizon* of a STL formula φ, denoted by hz_φ, is given in [31]. Informally, the horizon of a formula represents the smallest time duration from the current time into the future needed to decide the satisfaction and, equivalently, to compute the robustness, of any signal with respect to the formula. For example, $\varphi = \Diamond_{[0,t_1]} \Box_{[0,t_2]} (s > 0)$ has $\mathrm{hz}_\varphi = t_1 + t_2$.

Generalized Means. We describe generalizations of the usual arithmetic and geometric means, using a function g [32]. Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a continuous and injective function, $\mathbf{x} = [x_1, \ldots, x_d] \in \mathbb{R}^d_{\geq 0}$ a vector of positive real numbers, and $h : I \to \mathbb{R}_{\geq 0}$ a function over a domain $I \subseteq \mathbb{R}_{\geq 0}$. The generalized means of $\mathbf{x}$ and h associated with function g are defined as:

$$
F_g(\mathbf{x}) = g^{-1}\Big(\frac{1}{d}\sum_{i=1}^{d} g(x_i)\Big), \;\; F_g^I(h) = g^{-1}\Big(\frac{1}{|I|}\int_I g(h(t))\,\mathrm{d}t\Big) \tag{2}
$$

We are especially interested in the class of means called power means given by $g(x) = x^p$ and parameterized by the exponent $p \in \mathbb{R}$. For a real number p the power means of $\mathbf{x}$ and function h with exponent p are defined as:

$$M_p(\mathbf{x}) = M_p(x_1, \ldots, x_d) = \left(\frac{1}{d}\sum_{i=1}^{d} x_i^p\right)^{\frac{1}{p}} = d^{-\frac{1}{p}} \|\mathbf{x}\|_p, \quad (3)$$

$$M_p^I(h) = \left(\frac{1}{|I|}\int_I h^p(t)\,\mathrm{d}\nu\right)^{\frac{1}{p}} = |I|^{-\frac{1}{p}} \|h\|_p, \quad (4)$$

where ν is either the Lebesgue or discrete measure corresponding to continuous or discrete time, respectively. Based on the norm properties, both $M_p(\mathbf{x})$ and $M_p^I(h)$ belong to $\mathbb{R}_{\geq 0}$.

Similar to [32], we use the convention $M_p(x_1, \ldots, x_d) = 0$ for $p < 0$ and some $x_i = 0$, $i \in \{1, \ldots, d\}$. For the integral version, $M_p^I(h) = 0$ if h vanishes on a proper subset of I.

For $p = 1, 0, \infty, -\infty$, we recover the arithmetic mean, geometric mean, maximum and minimum, respectively [32, Sec.2.9]. For any $p, q \in \mathbb{R}$ with $p < q$ we have [32, Sec.2.9]:

$$M_p(\mathbf{x}) \leq M_q(\mathbf{x}). \quad (5)$$

$$M_{-\infty}(\mathbf{x}) \leq M_p(\mathbf{x}) \leq M_\infty(\mathbf{x}). \quad (6)$$

The two means are equal, $M_p(\mathbf{x}) = M_q(\mathbf{x})$ with $p \neq q$, if and only if all x_i are all equal [32, Sec.2.9].

Proposition 1: The following properties hold [32, Sec.2.9]:

1) *symmetry:* $M_p(P\mathbf{x}) = M_p(\mathbf{x})$ for all permutations P,
2) *fixed-point:* $M_p(\alpha \mathbf{1}_d) = \alpha$,
3) *absolutely scalable:* $M_p(\alpha\mathbf{x}) = \alpha M_p(\mathbf{x})$ for $\alpha \in \mathbb{R}_{\geq 0}$,
4) *monotonicity* in each argument,
5) *continuity* in each argument, and
6) *replacement:* $M_p(\mathbf{x}) = M_p(D_k m \mathbf{1}_d + (\mathbb{I}_d - D_k)\mathbf{x})$, where $D_k = diag([\mathbf{1}_k^T\ \mathbf{0}_{d-k}^T])$, $m = M_p(\mathbf{x}_{1:k})$, $1 \leq k \leq d$.

Replacement allows for block computation of mean, and divide and conquer evaluation algorithms.

Generalized means with function g are only guaranteed to satisfy properties 1, 2, and 5 from Prop. 1.

Boolean Algebra. The traditional robustness ρ is defined based on Boolean algebras. A Boolean algebra is a 6-uple $(\mathbb{D}, \sqcap, \sqcup, n, \rho_\top, \rho_\bot)$, where $\sqcap$, $\sqcup$, and n are the conjunction, disjunction, and negation operations on $\mathbb{D} = [-\alpha, \alpha]$; $\rho_\top = \alpha$, $\rho_\bot = -\alpha$ are the least and greatest elements, respectively [33]. For $a, b \in \mathbb{D}$, the identity and absorption laws are given by

$$a \sqcap \rho_\top = a, \qquad a \sqcup \rho_\bot = a, \qquad \text{(identity)} \quad (7)$$

$$a \sqcap (a \sqcup b) = a, \qquad a \sqcup (a \sqcap b) = a. \qquad \text{(absorption)} \quad (8)$$

Boolean algebras are also lattices. The partial order relation $\leq$ on $\mathbb{D}$ is induced by the conjunction. Formally, for $a, b \in \mathbb{D}$

$$a \leq b \quad \text{if and only if} \quad a = a \sqcap b \quad (\equiv\ b = a \sqcup b) \quad (9)$$

The traditional robustness is defined using $\sqcap = \min$, $\sqcup = \max$, and $n(x) = -x$. The order induced by $\min$ and $\max$ on $\mathbb{D}$ is the standard order relation $\leq$ on real numbers.

III. Motivation and Problem Formulations

The traditional robustness in Def. 1 has two main limitations, called *locality* and *masking*, which negatively impact the performance of optimization-based control and learning algorithms. These are discussed in Sec. III-A and motivate the class of robustness scores based on generalized means proposed in Sec. IV, which subsume other scores proposed in the literature [9], [18], [24], [28], [29], [34]. In this section, we also formalize the control synthesis problem for dynamical systems to meet STL specifications robustly.

A. Locality and Masking

In this section, we define two properties that motivated the development of the proposed robustness scores in the next section. The properties of locality and masking have impact on the performance of robustness scores in optimizations problems arising in control synthesis [35].

Let Φ be the set of all STL formulae and $\vartheta : \Phi \times \mathbb{S} \times \mathbb{T} \to \mathbb{R}$ a sound robustness score (the traditional robustness ρ defined in Def. 1 is a particular instance of ϑ).

Definition 2 (Locality): The robustness ϑ is said to be *local* if its value depends only on the value of signals at a single time instant. Formally, for every $\varphi \in \Phi$ with $\mathrm{hz}_\varphi > 0$, $\mathbf{s} \in \mathbb{S}$, and $t \in \mathbb{R}_{\geq 0}$, there exists $\tau \geq t$ such that $\vartheta(\varphi, \mathbf{s}, t) = \vartheta(\varphi, \mathbf{s}_\phi + \mathbf{s} \cdot \delta_\tau, t)$, where δ_τ is the Dirac delta function at τ, and $\mathbf{s}_\phi$ is a signal with zero robustness $\vartheta(\varphi, \mathbf{s}_\phi, t) = 0$.

Def. 2 implies that we can ignore the values of the signal $\mathbf{s}$ except at time τ.

Definition 3 (Masking): The robustness ϑ is said to be *masking* conjunction and disjunction if its values for $\wedge$ and $\vee$ operators depend on the robustness score of a particular subformula, respectively.

Formally, conjunction is masking if $\vartheta(\varphi_1 \wedge \varphi_2, \mathbf{s}, t) = \vartheta(\varphi_1, \mathbf{s}, t)$ whenever $\vartheta(\varphi_1, \mathbf{s}, t) \leq \vartheta(\varphi_2, \mathbf{s}, t)$, and disjunction is masking if $\vartheta(\varphi_1 \vee \varphi_2, \mathbf{s}, t) = \vartheta(\varphi_1, \mathbf{s}, t)$ whenever $\vartheta(\varphi_1, \mathbf{s}, t) \geq \vartheta(\varphi_2, \mathbf{s}, t)$, where $\varphi_1, \varphi_2 \in \Phi$, $\mathbf{s} \in \mathbb{S}$, and $t \in \mathbb{R}_{\geq 0}$.

Def. 3 means that we can ignore all but one operand in conjunction and disjunction formulae for computing the overall robustness score. Masking and locality of robustness may be desired in applications involving monitoring, where the portions of the formulae and signals that lead to largest violation need to be identified. However, in problems involving optimization such as control synthesis and learning, local and masking robustness scores impede the performance of optimization solvers. The masking and locality properties imply the existence of plateaus in the robustness landscape, and, thus, providing little gradient information. Moreover, in the other parts of the landscape, only a very small part of the formula and signal contributes gradient information.

Proposition 2: The traditional robustness score $\rho(\varphi, \mathbf{s}, t)$ is local and masking with respect to conjunction, and disjunction.

Proof: Locality and masking are consequences of the lattice structure of Boolean algebras (9). The masking condition for conjunction can be restated as $\vartheta(\varphi_1 \wedge \varphi_2, \mathbf{s}, t) = \min\{\vartheta(\varphi_1, \mathbf{s}, t), \vartheta(\varphi_2, \mathbf{s}, t)\}$, which matches traditional robustness (1). Similarly, disjunction can be restated in terms of $\max$, and, again, matches with (1). Locality can be shown using structural induction, where the induction step follows from the $\inf$ and $\sup$ operators that consider only the extreme value as the aggregate measure over their domains. ■

Example 2: Consider the car trajectory s_1 shown in Fig. 1 and formula φ_2 in Ex. 1. Although s_1 violates φ_2 at $t = 5, 6, 7$, the traditional robustness only considers the most violating time at $t = 6$. Therefore, we have $\rho(\varphi_2, s_1) = \rho(\varphi_2, s_1 \cdot \delta_6) = -2$ (locality). Next, assume the car has to follow a maximum speed limit given by STL formula $\varphi_3 = \Box_{[1,10]} v(t) \leq 5\frac{m}{s}$, and the maximum speed of the car given by trajectory s_1 is $5.3\frac{m}{s}$. Although s_1 violates φ_3 with $\rho(\varphi_3, s_1) = -0.3$, the traditional robustness of $\varphi_2 \wedge \varphi_3$ is only dependent on the robustness of φ_2, i.e., $\rho(\varphi_2 \wedge \varphi_3, s_1) = \rho(\varphi_2, s_1)$ (masking).

In Sec. IV, we formally define a class of robustness scores with tunable design parameters that can modulate the level of masking and locality.

B. Control Synthesis

Consider a dynamical system given by:

$$\mathbf{z}^{+}(t) = f(\mathbf{z}(t), \mathbf{u}(t)), \quad \mathbf{z}(0) = \mathbf{z}_0, \tag{10}$$

where $t \in \mathbb{T}$ is either discrete or continuous time, $\mathbf{z}^{+}(t)$ is either $\dot{\mathbf{z}}(t)$ or $\mathbf{z}(t+\delta t)$, $\mathbf{z}(t) \in \mathcal{Z} \subseteq \mathbb{R}^n$ is the state of the system, $\mathbf{u}(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input at time t, $\mathbf{z}_0 \in \mathcal{Z}$ is the initial state, and $f : \mathcal{Z} \times \mathcal{U} \to \mathcal{Z}$ is a Lipschitz continuous function representing the dynamics of the system. Given the initial state $\mathbf{z}_0$ and control signal $\mathbf{u}(t)$, the system trajectory $\mathbf{z} : \mathbb{T} \to \mathcal{Z}$ is generated using (10), and is denoted by $\mathbf{z}(\mathbf{z}_0, \mathbf{u})$. Consider a smooth cost function $J(\mathbf{z}(t), \mathbf{u}(t))$ and the system temporal specifications given by an STL formula φ over $\mathbf{z}$. Let ϑ be a sound robustness score.

The control synthesis problem can be formulated as determining a control policy $\mathbf{u}^*$ such that the system trajectory satisfies the STL specification φ with maximum robustness given by score ϑ and minimum cost:

$$\begin{aligned} \mathbf{u}^* = \operatorname{argmin}_{\mathbf{u}} \int_0^T J(\mathbf{u}(t), \mathbf{z}(t))\, \mathrm{d}\nu - \lambda \vartheta(\varphi, \mathbf{z}(\mathbf{z}_0, \mathbf{u})) \\ \text{s.t.} \quad (10),\ \vartheta(\varphi, \mathbf{z}(\mathbf{z}_0, \mathbf{u})) \geq \epsilon, \end{aligned} \tag{11}$$

where ν is either the Lebesgue or discrete measure corresponding to continuous or discrete time, respectively. The planning time horizon T must be chosen larger than the horizon hz_{φ} of φ such that the robustness $\vartheta(\varphi, \cdot)$ may be computed. The trade-off between the two objectives is captured by the weight $\lambda \geq 0$. The lower bound $\epsilon \geq 0$ gives the satisfaction margin as captured by the robustness score ϑ.

Note that since ϑ is assumed to be sound, the feasibility of (11) implies that the system trajectory $\mathbf{z}(\mathbf{z}_0, \mathbf{u}^*) \models \varphi$.

The main shortcoming of the traditional robustness score is that it only considers the robustness of the most satisfying or violating part of the specification, without taking into account satisfaction of the other parts. Consider the *eventually* operator, $\Diamond_I \varphi$, which is satisfied if φ is true at least once in the interval I. Traditional robustness would only capture the time with maximum satisfaction whereas we may want the robustness score to be affected by all the time points satisfying φ. In the following sections, we address this limitation by proposing a class of robustness functions based on generalized means that allows us to choose desired conservativeness of the score.

IV. Generalized Mean Robustness

Throughout the section, we assume signals are integrable with respect to both *Lebesgue* and discrete measures.

A. Robustness Definition

We first define functions associated with Boolean operators that will be used in the robustness definition. Let $\triangle : \mathbb{D}^d \to \mathbb{D}$ be the d−ary conjunction function defined over $\mathbf{x}$ using two power means with exponent parameters $p, q \in \mathbb{R} \cup \{\pm\infty\}$ as:

$$\triangle(\mathbf{x}) = \begin{cases} M_p(x_1, \ldots, x_d) & \text{if } \min(\mathbf{x}) > 0 \\ -M_q(-[x_1]_-, \ldots, -[x_d]_-) & \text{else} \end{cases} \tag{12}$$

More explicitly, the conjunction function $\triangle$ is determined by the power mean with exponent p if $\forall i :\ x_i > 0$, thus, it is positive ($M_p(\mathbf{x}) > 0$); otherwise by the power mean with exponent q defined over $x_i \leq 0$, thus, it is non-positive ($-M_q(-[\mathbf{x}]_-) \leq 0$). The branches in (12) enable us to define a sound robustness function (See Thm. 2).

Negation function $n : \mathbb{D} \to \mathbb{D}$ is given by $n(x) = -x$. The disjunction function $\triangledown$ is defined by DeMorgan's law, i.e., $\triangledown(x_1, \ldots, x_d) = n(\triangle(n(x_1), \ldots, n(x_d)))$, given by:

$$\triangledown(\mathbf{x}) = \begin{cases} M_q([x_1]_+, \ldots, [x_d]_+) & \text{if } \exists i :\ x_i > 0 \\ -M_p(-x_1, \ldots, -x_d) & \text{else} \end{cases}. \tag{13}$$

With a slight abuse of notation, we denote the conjunction function over a continuum of values from (4):

$$\triangle_I(h) = \begin{cases} M_p^I(h) & \text{if } \inf_{t \in I}\{h(t)\} > 0 \\ -M_q^I(-[h]_-) & \text{else} \end{cases}. \tag{14}$$

The disjunction function is extended to continuum domain in a similar way, and is denoted by $\triangledown_I(h)$. Lastly, we define the implication function $\triangleright : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ as $\triangleright(x, y) = \triangledown(-x, y)$. Next, we illustrate how the proposed functions relate to the conjunction function (min) used in the traditional robustness which is based on the Boolean lattice $(\mathbb{D}, \min, \max)$.

Remark 1: A weaker form of identity law (7) w.r.t. maximum true and minimum false hold for conjunction $\triangle(x, \rho_\perp) < 0$ and disjunction $\triangledown(x, \rho_\top) > 0$ for all $x \in \mathbb{D}$, respectively. Although $(\mathbb{D}, \triangle, \triangledown)$ is not a distributive lattice, i.e., Boolean algebra, it does satisfy the Kleene algebra condition: $\triangle(x, n(x)) \leq \triangledown(y, n(y))$, $\forall x, y \in \mathbb{D}$. An immediate consequence of (6) is that the conjunction and disjunction functions are bounded by the smallest and largest values of $\mathbf{x}$, respectively. Therefore, for all p and q, we have: $\triangle(x_1, \ldots, x_d) \geq M_{-\infty}(x_1, \ldots, x_d) = \min\{x_1, \ldots, x_d\}$ and $\triangledown(x_1, \ldots, x_d) \leq M_{\infty}(x_1, \ldots, x_d) = \max\{x_1, \ldots, x_d\}$.

Remark 2: [25, Prop. 1] The conjunction function cannot be sound, idempotent, and smooth simultaneously.

We define power mean robustness as the quantitative valuation of a property expressed by conjunction, disjunction, negation and implication functions being true or false.

Definition 4 (Power Mean Robustness): Let $\mathbf{s} : \mathbb{R}_{\geq 0} \to \mathbb{D}^n$, and φ be an STL formula. The power mean robustness of order (p, q) for signal $\mathbf{s}$ and formula φ at time t is

$$\begin{aligned} \eta_{p,q}(\top, \mathbf{s}, t) &:= \sup \mathbb{D},\ \ \eta_{p,q}(\mu, \mathbf{s}, t) := \ell(\mathbf{s}(t)) \\ \eta_{p,q}(\neg\varphi, \mathbf{s}, t) &:= n(\eta_{p,q}(\varphi, \mathbf{s}, t)) \\ \eta_{p,q}(\varphi_1 \wedge \varphi_2, \mathbf{s}, t) &:= \triangle\left(\eta_{p,q}(\varphi_1, \mathbf{s}, t), \eta_{p,q}(\varphi_2, \mathbf{s}, t)\right) \\ \eta_{p,q}(\varphi_1 \vee \varphi_2, \mathbf{s}, t) &:= \triangledown\left(\eta_{p,q}(\varphi_1, \mathbf{s}, t), \eta_{p,q}(\varphi_2, \mathbf{s}, t)\right) \\ \eta_{p,q}(\Box_I \varphi, \mathbf{s}, t) &:= \triangle_{t+I}\left(\eta_{p,q}(\varphi, \mathbf{s}, t')\right) \\ \eta_{p,q}(\Diamond_I \varphi, \mathbf{s}, t) &:= \triangledown_{t+I}\left(\eta_{p,q}(\varphi, \mathbf{s}, t')\right) \\ \eta_{p,q}(\varphi_1 \mathcal{U}_I \varphi_2, \mathbf{s}, t) &:= \triangledown_{t+I}\Big(\triangle\big(\eta_{p,q}(\varphi_2, \mathbf{s}, t'), \\ &\qquad \triangle_{[t,t']}\left(\eta_{p,q}(\varphi_1, \mathbf{s}, t'')\right)\big)\Big), \end{aligned} \tag{15}$$

where $p, q \in \mathbb{R} \cup \{\pm\infty\}$, $t' \in t + I$, and $t'' \in [t, t']$.

We denote $\eta_{p,q}(\varphi, \mathbf{s}, 0)$ by $\eta_{p,q}(\varphi, \mathbf{s})$.

In the following, we relate the class of power mean robustness scores to the traditional one, and characterize the properties of these extensions. We show that the class of power mean robustness is consistent with the semantics of STL.

Theorem 2 (Soundness): The power mean robustness is sound for all p and q, meaning that a strictly positive robustness shows a satisfying trajectory, and a strictly negative robustness shows a violating trajectory:

$$\begin{aligned} \eta_{p,q}(\varphi,\mathrm{s},t) > 0 &\iff \rho(\varphi,\mathrm{s},t) > 0 \implies \mathrm{s} \models \varphi, \\ \eta_{p,q}(\varphi,\mathrm{s},t) < 0 &\iff \rho(\varphi,\mathrm{s},t) < 0 \implies \mathrm{s} \not\models \varphi. \end{aligned} \tag{16}$$

Proof: We prove the theorem by structural induction over the formula φ. The *base case* corresponding to $\varphi \in \{\top,\bot,\mu\}$ is trivially true by definition from (15).

Let s be a signal. We have the following *induction cases*:

Negation: Let $\phi = \neg\varphi$ and $\eta_{p,q}(\phi,\mathrm{s},t) > 0$. From (15) we have $\eta_{p,q}(\varphi,\mathrm{s},t) < 0$, and by the induction hypothesis $\mathrm{s} \not\models \varphi$. Thus, $\mathrm{s} \models \phi$. Similarly, for $\eta_{p,q}(\phi,\mathrm{s},t) < 0$ we get $\mathrm{s} \not\models \phi$.

Conjunction: Let $\phi = \varphi_1 \wedge \varphi_2$ and $\eta_{p,q}(\phi,\mathrm{s},t) > 0$. Assume that one or both $\eta_{p,q}(\varphi_i,\mathrm{s},t) < 0$, $i = 1,2$, then from (15), (12) we get $\eta_{p,q}(\phi,\mathrm{s},t) = -M_q(-[\eta_{p,q}(\varphi_1,\mathrm{s},t)]_-, -[\eta_{p,q}(\varphi_2,\mathrm{s},t)]_-)$ which is negative and contradicts the assumption that $\eta(\phi,\mathrm{s},t) > 0$. It follows that $\eta_{p,q}(\varphi_i,\mathrm{s},t) > 0$, $i = 1,2$. By the induction hypothesis $\mathrm{s} \models \varphi_i$, $i = 1,2$, and thus $\mathrm{s} \models \phi$. The same analysis is applied for the case where $\eta_{p,q}(\phi,\mathrm{s},t) < 0$. By way of contradiction, assume $\eta_{p,q}(\varphi_i,\mathrm{s},t) > 0$, $i = 1,2$. From (15), (12) it follows that $\eta_{p,q}(\phi,\mathrm{s},t) = M_p(\eta_{p,q}(\varphi_1,\mathrm{s},t), \eta_{p,q}(\varphi_2,\mathrm{s},t)) > 0$ which is a contradiction. Thus, we have either $\eta_{p,q}(\varphi_1,\mathrm{s},t) < 0$ or $\eta_{p,q}(\varphi_2,\mathrm{s},t) < 0$ or both. Again by the induction hypothesis $\mathrm{s} \not\models \varphi_1$ or $\mathrm{s} \not\models \varphi_2$, therefore, $\mathrm{s} \not\models \phi$.

Disjunction: Follows similarly to *conjunction* case.

Globally: Let $\phi = \Box_I \varphi$, and $\eta_{p,q}(\phi,\mathrm{s},t) > 0$. By way of contradiction, assume that there is $t'' \in t + I$ such that $\eta_{p,q}(\varphi,\mathrm{s},t'') < 0$, then from (15), (14) we get $\eta_{p,q}(\phi,\mathrm{s},t) = -M_q^I(-[\eta_{p,q}(\varphi,\mathrm{s},t')]_-) < 0$ which contradicts the assumption $\eta_{p,q}(\phi,\mathrm{s},t) > 0$. It follows that $\eta_{p,q}(\varphi,\mathrm{s},t') > 0$, $\forall t' \in t + I$. By the induction hypothesis $\mathrm{s}(t') \models \varphi$, $\forall t' \in t + I$, and thus $\mathrm{s} \models \phi$. Similarly, for the case $\eta_{p,q}(\phi,\mathrm{s},t) < 0$, assume that for all $t' \in t + I$, $\eta_{p,q}(\varphi,\mathrm{s},t') > 0$. From (15), (14) we have $\eta_{p,q}(\phi,\mathrm{s},t) = M_p^I(\eta_{p,q}(\varphi,\mathrm{s},t')) > 0$ which is a contradiction. Thus, we have $\eta_{p,q}(\varphi,\mathrm{s},t'') < 0$ for some $t'' \in t+I$. Again by the induction hypothesis $\mathrm{s}(t'') \not\models \varphi$, thus $\mathrm{s} \not\models \phi$.

Eventually and *Until:* Follow similarly to the *globally* case. ■

Soundness can be viewed as a sign consistency between power mean and traditional robustness scores. The next result shows how the magnitude of the score changes with the two parameters of the proposed class.

Theorem 3: For any STL formula φ and signal s, we have $|\eta_{p,q}(\varphi,\mathrm{s},t)| < |\eta_{p',q'}(\varphi,\mathrm{s},t)|$, for any $p < p'$ and $q < q'$.

Proof: The claim follows from (5) and Def. 4 by structural induction over the formula φ. ■

As opposed to traditional robustness, the proposed class is maximally satisfied (violated) if this also holds for all its sub-formulae.

Proposition 3: Let s be a signal, ϕ a STL formula with horizon I and sub-formulae φ_i, and $p,q \in \mathbb{R}$. If $\eta_{p,q}(\phi,\mathrm{s},t) = \rho_\top$, then $\eta_{p,q}(\varphi_i,\mathrm{s},t) = \rho_\top$ for all sub-formulae φ_i and times $t \in I$ as given by (15). If $\eta_{p,q}(\phi,\mathrm{s},t) = \rho_\bot$, then $\eta_{p,q}(\varphi_i,\mathrm{s},t) = \rho_\bot$ for all sub-formulae φ_i and times $t \in I$ in (15).

Proof: The proof is similar to Thm. 2 and holds for any real value p,q [32, Sec.2.9]. ■

The next result shows that the power mean robustness of finite order ($|p|,|q| < \infty$) avoids the masking and locality issues of the traditional robustness in (1).

Theorem 4: For $p,q \in \mathbb{R}$, the robustness $\eta_{p,q}$ is non-local, and non-masking with respect to conjunction and disjunction.

Proof: First, consider the non-masking property. Let $\varphi = \varphi_1 \wedge \varphi_2$, $\mathrm{s} \in \mathbb{S}$, and $t \in \mathbb{R}_{\geq 0}$, such that $\eta_{p,g}(\varphi_1,\mathrm{s},t) \leq \eta_{p,g}(\varphi_2,\mathrm{s},t)$. Denote $\zeta_i = \eta_{p,g}(\varphi_i,\mathrm{s},t)$, $i \in \{1,2\}$.

Assume that $\zeta_1 > 0$. By Def. 4, $\eta_{p,q}(\varphi,\mathrm{s},t) = M_p(\zeta_1,\zeta_2)$. Thus, $\min_i \zeta_i < \eta_{p,q}(\varphi,\mathrm{s},t) < \max_i \zeta_i$, $i \in \{1,2\}$ unless $\zeta_1 = \zeta_2$ due to (6). Since the property has to hold for all formulae φ_i, $i \in \{1,2\}$, signals s, and times $t \geq 0$, it follows that the class of power mean robustness is non-masking with respect to conjunction. Non-masking for disjunction follows similarly.

Next, we show that $\eta_{p,q}$ is non-local. Consider $\varphi = \Diamond_{[0,T]} s \geq 0$, and a continuous scalar signal $\mathfrak{s}$ taking positive values. It follows that $\inf_{[0,T]} \mathfrak{s}(t) > 0$, and by Def. 4 $\eta_{p,q}(\varphi,\mathfrak{s},0) = M_p^{[0,T]}(\mathfrak{s}) = \left(\frac{1}{T}\int_0^T (\mathfrak{s}(t))^p \,\mathrm{d}t\right)^{\frac{1}{p}}$. The latter identity shows that $\eta_{p,q}(\varphi,\mathfrak{s},0)$ depends on all values of the signal in the time domain $[0,T]$. Thus, the generalized mean robustness is non-local. ■

Similar to traditional robustness, the proposed class of robustness satisfies the following properties.

Proposition 4: The power mean robustness for all orders $p,q \in \mathbb{R} \cup \{\pm\infty\}$ holds the following properties:

1) *Commutativity:* $\eta_{p,q}(\varphi_1 \wedge \varphi_2,\mathrm{s}) = \eta_{p,q}(\varphi_2 \wedge \varphi_1,\mathrm{s})$
2) *Idempotence:* $\eta_{p,q}(\varphi \wedge \varphi,\mathrm{s}) = \eta_{p,q}(\varphi,\mathrm{s})$
3) *Absolutely scalable:* $\eta_{p,q}(\varphi,\alpha\mathrm{s}) = \alpha \cdot \eta_{p,q}(\varphi,\mathrm{s})$, $\alpha \in \mathbb{R}_{\geq 0}$,
4) *Monotonicity:* $\eta_{p,q}(\varphi_1 \wedge \varphi_2,\mathrm{s}) \leq \eta_{p,q}(\varphi_3 \wedge \varphi_4,\mathrm{s})$, $\forall \varphi_i$ where $\eta_{p,q}(\varphi_1,\mathrm{s}) \leq \eta_{p,q}(\varphi_3,\mathrm{s})$, $\eta_{p,q}(\varphi_2,\mathrm{s}) \leq \eta_{p,q}(\varphi_4,\mathrm{s})$
5) *Continuity* of $\eta_{p,q}(\varphi_1 \wedge \varphi_2,\mathrm{s})$ in $\eta_{p,q}(\varphi_1,\mathrm{s})$ and $\eta_{p,q}(\varphi_2,\mathrm{s})$.

Proof: The claims follow from the symmetry, fixed-point, absolutely scalable, monotonicity and continuity property of power mean in Prop. 1. Similarly, the same properties hold for disjunction $\vee$ operator. ■

Similar to traditional robustness, the proposed class of robustness satisfies the following logical properties.

Proposition 5 (Rules of Inference): The following hold for power mean robustness of all orders $p,q \in \mathbb{R} \cup \{\pm\infty\}$:

1) *Law of non-contradiction:* $\eta_{p,q}(\varphi \wedge \neg\varphi,\mathrm{s}) < 0$, $\forall\varphi$ where $\eta_{p,q}(\varphi,\mathrm{s}) \neq 0$,
2) *Law of excluded middle:* $\eta_{p,q}(\varphi \vee \neg\varphi,\mathrm{s}) > 0$, $\forall\varphi$ where $\eta_{p,q}(\varphi,\mathrm{s}) \neq 0$,
3) *Double negation:* $\eta_{p,q}(\neg(\neg\varphi),\mathrm{s}) = \eta_{p,q}(\varphi,\mathrm{s})$, $\forall\varphi$
4) *DeMorgan's law:* $\eta_{p,q}(\varphi_1 \vee \varphi_2,\mathrm{s}) = \eta_{p,q}(\neg(\neg\varphi_1 \wedge \neg\varphi_2),\mathrm{s})$, $\eta_{p,q}(\Box_I\varphi,\mathrm{s}) = \eta_{p,q}(\neg\Diamond_I\neg\varphi,\mathrm{s})$.
5) *Modus ponens:* if $\eta_{p,q}(\varphi_1 \Rightarrow \varphi_2,\mathrm{s}) > 0$ and $\eta_{p,q}(\varphi_1,\mathrm{s}) > 0$ then $\eta_{p,q}(\varphi_2,\mathrm{s}) > 0$.

Proof: The proof follows from the properties of Boolean algebras and soundness of Def. 4 and is omitted. ■

In the following, we define the generalized mean robustness.

Definition 5 (Generalized Mean Robustness): Let s be a signal, and φ an STL formula. The generalized mean robustness $\eta_{c,g}$ with continuous and injective functions c,g for signal s and formula φ at time t is recursively defined similar

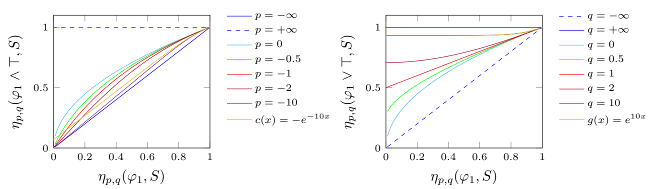

Fig. 2. Power mean robustness for varying values of p in $\eta_{p,q}(\varphi_1 \wedge \top, \mathbf{s})$ and q in $\eta_{p,q}(\varphi_1 \vee \top, \mathbf{s})$

to (15), where the d-ary and continuum conjunction functions are defined based on generalized means as

$$\triangle(\mathbf{x}) = \begin{cases} F_c(x_1, \dots, x_d) & \text{if } \min(\mathbf{x}) > 0 \\ -F_g(-[x_1]_-, \dots, -[x_d]_-) & \text{else} \end{cases}$$
$$\triangle_I(h) = \begin{cases} F_c^I(h) & \text{if } \inf_{t \in I}\{h(t)\} > 0 \\ -F_g^I(-[h]_-) & \text{else} \end{cases} \quad (17)$$

The negation $n : \mathbb{D} \to \mathbb{D}$ is given by $n(x) = -x$ and disjunction function is defined by DeMorgan's law.

Remark 3: Generalized mean robustness is guaranteed to satisfy Thm. 3, Prop. 3, Prop. 5, properties 1, 2, and 5 from Prop. 4. Satisfaction of other properties and theorems depends on the functions c and g.

As before, if we set $c(x) = |x|^p$ and $g(x) = |x|^q$, then we recover the power mean robustness in Def. 4.

B. Performance Properties

The parameters p, q associated with power mean are considered as design parameters that can be used to tune how conservative the robustness score is. Assume $\mathbb{D} = [-1, 1]$, i.e., $\rho_\top = 1, \rho_\bot = -1$. Consider the conjunction operator $\varphi_1 \wedge \varphi_2$, and the case where conjunction of a satisfying signal $\mathbf{s}$ with $\varphi_2 = \top$ is evaluated, i.e., $\eta_{p,q}(\varphi_1, \mathbf{s}) > 0$, and $\eta_{p,q}(\varphi_2, \mathbf{s}) = 1$. Fig. 2 shows how changing values of p changes the power mean robustness $\eta_{p,q}(\varphi_1 \wedge \top, \mathbf{s})$. Note that for more negative values of p, $\eta_{p,q}(\varphi_1 \wedge \top, \mathbf{s})$ is closer to the traditional robustness $\rho(\varphi_1 \wedge \top, \mathbf{s}) = \min(\rho(\varphi_1, \mathbf{s}), 1)$ corresponding to $p = -\infty$ (See sec. IV-C). Next, consider the disjunction operator $\varphi_1 \vee \varphi_2$ with $\varphi_2 = \top$. Fig. 2 shows how changing values of q changes the power mean robustness $\eta_{p,q}(\varphi_1 \vee \top, \mathbf{s})$. Note that for more positive values of q, $\eta_{p,q}(\varphi_1 \wedge \top, \mathbf{s})$ is closer to the traditional robustness $\rho(\varphi_1 \vee \top, \mathbf{s}) = \max(\rho(\varphi_1, \mathbf{s}), 1)$ corresponding to $q = \infty$ (See Sec. IV-C). It can be seen that the power mean robustness (for $p, q \in \mathbb{R}$) and the generalized mean robustness for the chosen c, g can tune the locality and masking of the score. Moreover, both power and generalized mean robustness scores are bounded by the robustness for $p = -\infty, q = \infty$ as in Thm. 3. The maximum robustness is achieved at $\eta_{p,q}(\varphi_1, \mathbf{s}) = \eta_{p,q}(\varphi_2, \mathbf{s}) = 1$ independent of p, q according to Prop. 3.

Therefore, similar to the traditional robustness (minimum and maximum functions), power mean robustness with $p, q = -\infty, \infty$ only considers the most critical time or sub-formula and is useful for monitoring or control in safety-critical applications. On the other hand, by changing the values for p, q of the power mean robustness or the functions c, g in generalized mean robustness, we can determine the level of contribution of all the sub-formulae and times in the overall satisfaction or violation score of a specification.

Example 3: For STL formulae φ_1 and φ_2 in Ex. 1, we find the power mean robustness $\eta_{p,q}$ for trajectories s_1 and s_2 shown in Fig. 1. We first discuss the satisfaction case for φ_1. In the table below, power mean robustness $\eta_{p,q}$ is calculated from (15), (12) for different values of p. As discussed earlier, for $p = -\infty$ (similar to the traditional robustness), power mean robustness considers the most extreme time point (s_1 at $t \in \{1, 2, 9, 10\}$, s_2 at $t \in \{1, 10\}$) leading to an equal robustness for both signals. However, for $p \in \mathbb{R}$, robustness is based on the power mean of all the satisfying time points, and can distinguish the performance of the shown trajectories. Same analysis holds for the violation case considering φ_2. Although s_2 violates φ_2 for a longer duration than s_1 (s_1 violates φ_2 at $t \in \{5, 6, 7\}$ while s_2 violates it at $t \in \{3, 4, \dots, 8\}$), by only considering the most violating time ($s_1(6)$ and $s_2(t)$, $t \in \{4, \dots, 8\}$), the power mean robustness for $q = \infty$ for both trajectories is equal. The table shows power mean robustness $\eta_{p,q}$ calculated for different values of q. Note that for more negative values of p and more positive values of q the power mean robustness $\eta_{p,q}$ is closer to the traditional robustness.

	p for φ_1					q for φ_2			
	0	-1	-2	-20	$-\infty$	1	2	20	$+\infty$
S_1	3.21	2.94	2.72	2.09	2.0	-0.40	-0.77	-1.78	-2.0
S_2	4.36	3.94	3.53	2.16	2.0	-1.1	-1.45	-1.93	-2.0

Properties 1 (Smoothness and Gradient): The power mean robustness $\eta_{p,q}(\varphi, \mathbf{s}, t)$ is smooth in $\mathbf{s} \in \mathbb{D}^n$ everywhere except on the satisfaction boundaries where its sign changes. Moreover, the gradient of $\eta_{p,q}$ with respect to the elements of $\mathbf{s}$ that are part of φ's predicates is non-zero wherever it is smooth (non-local and non-mask). This property follows by smoothness and non-zero gradient of the conjunction $\triangle$ and disjunction $\triangledown$ functions on $\left(\mathbb{D} \setminus \{0\}\right)^d$, and negation n on $\mathbb{D}$. The cases for the *globally*, *eventually* and *until* operators follow similarly. The generalized mean robustness is smooth if the functions c and g are smooth, and the gradient depends on the functions c and g.

C. Generalized Mean Robustness as a Unified Class

In this section, we show that the generalized mean robustness defined here is a unified class of robustness scores. Specifically, it encompasses robustness scores in the literature including traditional robustness [10], average robustness [26], smooth robustness [18] and smooth cumulative robustness [27], and arithmetic-geometric mean (AGM) [24] and arithmetic-geometric integral mean (AGIM) robustness [29].

To show this, we start with the traditional robustness [10]. For $p = -\infty$, $q = \infty$, the power mean robustness is defined as:

$$\triangle(\mathbf{x}) = \begin{cases} M_{-\infty}(\mathbf{x}) = \min(\mathbf{x}) & \text{if } \min(\mathbf{x}) > 0 \\ -M_\infty(-[\mathbf{x}]_-) = -\max(-[\mathbf{x}]_-) = & \text{else} \end{cases}$$

Note that $-\max(-[\mathbf{x}]_-) = \min([\mathbf{x}]_-) = \min(\mathbf{x})$ due to the branch condition. Therefore, $\triangle(\mathbf{x})$ becomes $\min(\mathbf{x})$ which is the conjunction function in traditional robustness. Similarly, we can show that $\triangledown(\mathbf{x}) = \max(\mathbf{x})$. Thus, the power mean robustness with $p = -\infty$, $q = \infty$ in (15) is the same as traditional robustness in (1).

Next, consider the AverageSTL robustness [26], which uses ρ^+ and ρ^- for satisfying and violating parts of the signal to define new averaged operators. Averaged-eventually and Averaged-always can be formulated as power mean robustness with $p=1$, $q=1$. Consider the power mean robustness for $\diamond$ operator with $p=1$, $q=1$:

$$\triangledown_I(h) = \begin{cases} M_1^I([h]_+) = \frac{1}{|I|}\int_I [h(t)]_+ \,\mathrm{d}t & \text{if } \sup_{t\in I}\{h(t)\} > 0 \\ -M_1^I(-h) = -\frac{1}{|I|}\int_I -h(t)\,\mathrm{d}t & \text{else} \end{cases}$$

Note that the first branch is similar to $\rho^+_{AverageSTL}$ in definition of averaged-eventually in [26]. Based on the branch condition, we have $-\frac{1}{|I|}\int_I -h(t)\,\mathrm{d}t = \frac{1}{|I|}\int_I h(t)\,\mathrm{d}t = \frac{1}{|I|}\int_I [h(t)]_- \,\mathrm{d}t$ and the second branch is also the same as $\rho^-_{AverageSTL}$ in [26]. Averaged-always can be obtained similarly, therefore, we can build the average robustness [26] from the power mean robustness $\eta_{1,1}$.

We can build the smooth robustness in [18] from the generalized mean robustness. Smooth robustness $\tilde{\rho}$ uses approximations of $\max$ and $\min$ functions defined based on Logarithmic Sum of Exponentials (LSE) [18]: $\mathrm{LSE}_\beta(x_1,\dots,x_d) = \frac{1}{\beta}\log\left(\sum_{i=1}^d e^{\beta x_i}\right)$, where $\mathrm{LSE}_\beta(\mathbf{x})$ gets close to $\max(\mathbf{x})$ as $\beta \to \infty$. For $g(x) = e^{\beta x}$, the generalized mean robustness $F_g(x_1,\dots,x_d)$ is a constant shifted LSE function: $F_g(x_1,\dots,x_d) = \mathrm{LSE}_\beta(x_1,\dots,x_d) - \frac{\log(d)}{\beta}$. Let $c(x) = -e^{-\beta x}$, $g(x) = e^{\beta x}$. We define the conjunction function using generalized means as:

$$\triangle(\mathbf{x}) = \begin{cases} F_c(\mathbf{x}) = -\mathrm{LSE}_\beta(-\mathbf{x}) + \frac{\log(d)}{\beta} & \min(\mathbf{x}) > 0 \\ -F_g(-[\mathbf{x}]_-) = -\mathrm{LSE}_\beta(-[\mathbf{x}]_-) + \frac{\log(d)}{\beta} & \text{else} \end{cases}$$

As $\beta \to \infty$, we have:

$$\lim_{\beta\to\infty} \triangle(\mathbf{x}) = \begin{cases} \lim_{\beta\to\infty} -\mathrm{LSE}_\beta(-\mathbf{x}) & \text{if } \min(\mathbf{x}) > 0 \\ \lim_{\beta\to\infty} -\mathrm{LSE}_\beta(-[\mathbf{x}]_-) & \text{else} \end{cases}$$

which is similar to the conjunction function in robustness $\tilde{\rho}$ in [18]. Therefore, the smooth approximation robustness can be reconstructed from the generalized mean robustness and we get $\lim_{\beta\to\infty}\eta_{c,g} = \lim_{\beta\to\infty}\tilde{\rho} = \rho$. Note that this definition for real values of β is not monotone or scalable. The smooth cumulative robustness [27] is defined similar to the average robustness and smooth approximation robustness.

Finally, we show that the AGM and AGIM robustness in [24], [29] can be defined using the generalized mean robustness. Let $c(x) = \ln(1+x)$ and $g(x) = (1+x)$. The generalized means for c and g functions are $F_c(\mathbf{x}) = \exp\left(\frac{1}{d}\sum_{i=1}^d \ln(1+x_i)\right) - 1 = \left(\prod_{i=1}^d (1+x_i)\right)^{\frac{1}{d}} - 1$ and $F_g(\mathbf{x}) = \left(\frac{1}{d}\sum_{i=1}^d (1+x_i)\right) - 1 = \frac{1}{d}\sum_{i=1}^d x_i$ The conjunction function using the generalized means is

$$\triangle(\mathbf{x}) = \begin{cases} F_c(\mathbf{x}) = \left(\prod_{i=1}^d (1+x_i)\right)^{\frac{1}{d}} - 1 & \text{if } \forall i:\ x_i > 0 \\ -F_g(-[\mathbf{x}]_-) = \frac{1}{d}\sum_{i=1}^d [x_i]_- & \text{else} \end{cases}$$

which matches the AGM conjunction function in [24]. The disjunction function and other temporal operators for the AGM robustness η_{AGM} are derived similarly.

The same procedure can be applied to recover the AGIM robustness [29] in continuous-time based on the continuous definition of generalized means over continuum values, i.e., $F_c^I(h) = \exp\left(\frac{1}{|I|}\int_I \ln(1+h(t))\,\mathrm{d}\nu\right) - 1$ and $F_g^I(h) = \left(\frac{1}{|I|}\int_I (1+h(t))\,\mathrm{d}\nu\right) - 1 = \frac{1}{|I|}\int_I h(t)\,\mathrm{d}\nu$.

V. Control Synthesis

The power mean robustness is a unified class of robustness including the well known robustness measures [10], [18], [24], [26], [28]. Therefore, all the previously defined robustness optimization approaches including heuristics, and gradient-based methods can be used to solve (11). For $p,q \in \{-\infty, 1, \infty\}$, MILPs can be used. The gradient-based methods are suitable for general nonlinear dynamical systems and nonlinear STL predicates, and are shown to be scalable as the complexity of the formula or its horizon grows.

We first discuss our framework to solve (11) for a finite discrete-time system of length T. We assume the control input to be synthesized is defined as a sequence $\mathbf{u} = \{\mathbf{u}(0)\mathbf{u}(1)\dots\mathbf{u}(T-1)\}$. To initialize the gradient-based optimization algorithm, a random input sequence $\mathbf{u}^0 \in \mathcal{U}$ is generated, and the resulting trajectory $\mathbf{z}(\mathbf{z}_0,\mathbf{u})$ starting from initial state z_0 is found from the system dynamics (10), which may violate the STL specification. The optimization procedure recursively solves (11) to find optimal control policy $\mathbf{u}^* = \{\mathbf{u}^*(0)\mathbf{u}^*(1)\dots\mathbf{u}^*(T-1)\}$ which maximizes the generalized mean robustness for the given STL formula φ with respect to the system execution $\mathbf{z}(\mathbf{z}_0,\mathbf{u})$ and minimizes the cost. For a continuous-time system, same optimization algorithm considering a zeroth-order hold design can be applied [29]. Due to non-smoothness in $\eta_{p,q}$ at the satisfaction boundaries, we can use non-smooth optimization algorithms such as stochastic gradient methods or Alternating Direction Method of Multipliers [36], [37]. In this paper, we solve the optimization problem (11) using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) method. BFGS is shown to have acceptable performance for non-smooth optimization instances [38]. Another way to enhance the optimization is to first maximize a smooth robustness to have $\vartheta \ge \epsilon$, where ϑ is any smooth robustness measure and ϵ is chosen such that soundness holds, i.e., $\vartheta \ge \epsilon \Rightarrow \eta_{p,q} > 0$. We can then use this satisfying solution to initialize the generalized mean robustness optimization. To further improve the optimization convergence, we replace $[f]_+$ and $[f]_-$ with their LSE_β smooth approximations.

All simulations are implemented in Python running on an iMac with 3.3GHz Intel Core i5 CPU 32GB RAM. BFGS algorithm and SLSQP from Scipy package are used for optimization [39].

Example 4: Consider a dynamical system given by $x(t+1) = x(t) + u_x(t)$, $y(t+1) = y(t) + u_y(t)$, and a STL formula φ_3 in which desired regions are required to be sequentially visited within the associated deadlines:

$$\varphi_3 = \diamond_{[1,6]}\mathit{Blue} \wedge \diamond_{[7,15]}\mathit{Green} \wedge \diamond_{[16,18]}\square_{[0,2]}\mathit{Red} \wedge \square_{[1,20]}\neg\mathit{Obstacle} \wedge \square_{[1,20]}\mathit{Boundary}. \quad (18)$$

Each region is formulated as conjunction over the states in the 2-dimensional plane. For instance, *Blue* : $7 \le x \wedge x \le 9 \wedge 1 \le y \wedge y \le 3$. $\mathbf{z}(t) = [x(t), y(t)]$ is the state indicating position and orientation with *Boundary* $\mathcal{Z} = [0,10]^2$ and initial state $\mathbf{z}_0 = [1,1]$. $\mathbf{u}(t) = [u_x(t), u_y(t)]$ is the input vector with $\mathcal{U} = [-3,3]^2$. The formula requires the system to "*Eventually* visit *Blue* between $[1,6]$ steps *and eventually* visit *Green* between $[7,15]$ steps *and eventually* visit *Red* between $[16,18]$ and *Always* stay in *Red* for 2 steps *and Always* avoid *Obstacle* and *Always* stay inside the *boundary*".

Fig. 3 shows trajectories satisfying φ_3 and optimizing cost in (11) with $T = 20$, $J = \frac{1}{2}\sum_{i=1}^{T-1}\|\mathbf{u}(t)\|^2$ and $\lambda = 100$, considering 3 different generalized mean robustness scores,

(a) $p = -\infty$, $q = \infty$, (b) $c(x) = -e^{-\beta x}$, $g(x) = e^{\beta x}$ where $\beta = 10$, and (c) $p = 0$, $q = 1$, achieved up to the same termination criteria. The trajectory S_5 found by maximizing $\eta_{0,1}$ visits each region at multiple time points (maximum temporal satisfaction), and visits the centers of all (including the non-symmetrical) regions since these points correspond to maximum space satisfaction.

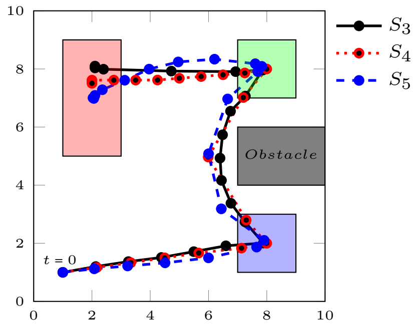

Fig. 3. Optimal trajectories satisfying φ_3 optimizing different generalized mean robustness scores. (a) S_3 : $p = -\infty$, $q = \infty$, (b) S_4: $g(x) = -e^{-\beta x}$, $g(x) = e^{\beta x}$, and (c) S_5: $p = 0, q = 1$. $\eta_{-\infty,\infty}(\varphi_3, S_3) = \eta_{-\infty,\infty}(\varphi_3, S_4) = \eta_{-\infty,\infty}(\varphi_3, S_5)$, $\eta_{c,g}(\varphi_3, S_3) = \eta_{c,g}(\varphi_3, S_4) = \eta_{c,g}(\varphi_3, S_5)$, $\eta_{0,1}(\varphi_3, S_3) < \eta_{0,1}(\varphi_3, S_4) < \eta_{0,1}(\varphi_3, S_5)$.

VI. Conclusion

We defined a general unified class of robustness scores based on power means and generalized means, parameterized by two continuous values which act as design parameters. We demonstrated the advantages of this new definition in theory and through empirical analysis in simulated trials, and proposed algorithms for evaluation, and control synthesis of systems under temporal requirements. Future work will focus on improving the control synthesis problem by using LP-norm approximations to smooth the robustness for an approximate solution, and to provide a toolbox based on updated SQP or BFGS methods with Gradient Sampling (GS).

References

[1] E. M. Clarke, E. A. Emerson, and A. P. Sistla, "Automatic verification of finite-state concurrent systems using temporal logic specifications," *ACM Tran. on Prog. Lang. and Syst.*, vol. 8, no. 2, pp. 244–263, 1986.

[2] C. Baier and J. Katoen, *Principles of model checking*. MIT Press, 2008.

[3] C. Belta, B. Yordanov, and E. A. Gol, *Formal methods for discrete-time dynamical systems*. Springer, 2017, vol. 89.

[4] P. Tabuada, *Verification and control of hybrid systems: a symbolic approach*. Springer Science & Business Media, 2009.

[5] R. Alur, T. A. Henzinger, and E. D. Sontag, *Hybrid systems III: verification and control*. Springer, 1996, vol. 3.

[6] A. Pnueli, "The temporal logic of programs," in *18th Annual Symposium on Foundations of Computer Science*, Oct 1977, pp. 46–57.

[7] R. Koymans, "Specifying real-time properties with metric temporal logic," *Real-time systems*, vol. 2, no. 4, pp. 255–299, 1990.

[8] C.-I. Vasile, D. Aksaray, and C. Belta, "Time window temporal logic," *Theoretical Computer Science*, vol. 691, pp. 27–54, 2017.

[9] O. Maler and D. Nickovic, "Monitoring temporal properties of continuous signals," in *Formal Techniques, Modelling and Analysis of Timed and Fault-Tolerant Systems*. Springer, 2004, pp. 152–166.

[10] A. Donzé and O. Maler, "Robust satisfaction of temporal logic over real-valued signals," in *International Conference on Formal Modeling and Analysis of Timed Systems*. Springer, 2010, pp. 92–106.

[11] V. Raman, A. Donzé, M. Maasoumy, R. M. Murray, A. Sangiovanni-Vincentelli, and S. A. Seshia, "Model predictive control with signal temporal logic specifications," in *Conference on Decision and Control*, Dec 2014, pp. 81–87.

[12] V. Raman, A. Donzé, D. Sadigh, R. M. Murray, and S. A. Seshia, "Reactive synthesis from signal temporal logic specifications," in *Intl Conf on Hybrid Systems: Computation and Control*, 2015, pp. 239–248.

[13] C. Belta and S. Sadraddini, "Formal methods for control synthesis: An optimization perspective," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 2, pp. 115–140, 2019.

[14] W. Liu, N. Mehdipour, and C. Belta, "Recurrent neural network controllers for signal temporal logic specifications subject to safety constraints," *IEEE Control Systems Letters*, 2021.

[15] N. Mehdipour, D. Briers, I. Haghighi, C. M. Glen, M. L. Kemp, and C. Belta, "Spatial-temporal pattern synthesis in a network of locally interacting cells," in *Conference on Decision and Control*. IEEE, 2018.

[16] S. M. LaValle and J. J. Kuffner Jr, "Randomized kinodynamic planning," *The Intl J of Robotics Research*, vol. 20, no. 5, pp. 378–400, 2001.

[17] S. Saha and A. A. Julius, "An MILP approach for real-time optimal controller synthesis with metric temporal logic specifications," in *American Control Conference (ACC)*. IEEE, 2016, pp. 1105–1110.

[18] Y. V. Pant, H. Abbas, and R. Mangharam, "Smooth operator: Control using the smooth robustness of temporal logic," in *IEEE Conference on Control Technology and Applications (CCTA)*, 2017, pp. 1235–1240.

[19] X. Li, Y. Ma, and C. Belta, "A policy search method for temporal logic specified reinforcement learning tasks," in *American Control Conference (ACC)*. IEEE, 2018, pp. 240–245.

[20] A. Rodionova, E. Bartocci, D. Nickovic, and R. Grosu, "Temporal logic as filtering," in *Intl Conf on Hybrid Systems: Computation and Control*, ser. HSCC '16. NY, USA: ACM, 2016, p. 11–20.

[21] R. Bloem, H. Chockler, M. Ebrahimi, and O. Strichman, "Specifiable robustness in reactive synthesis," *Formal Methods in System Design*, vol. 60, no. 2, pp. 259–276, 2022.

[22] T. Anevlavis, M. Philippe, D. Neider, and P. Tabuada, "Being correct is not enough: Efficient verification using robust linear temporal logic," *ACM Trans. Comput. Logic*, vol. 23, no. 2, jan 2022.

[23] D. Kamale, E. Karyofylli, and C. I. Vasile, "Automata-based Optimal Planning with Relaxed Specifications," in *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, 2021, pp. 6525–6530.

[24] N. Mehdipour, C.-I. Vasile, and C. Belta, "Arithmetic-geometric mean robustness for control from signal temporal logic specifications," in *American Control Conference (ACC)*. IEEE, 2019, pp. 1690–1695.

[25] P. Varnai and D. V. Dimarogonas, "On robustness metrics for learning stl tasks," in *American Control Conference (ACC)*, 2020, pp. 5394–5399.

[26] T. Akazaki and I. Hasuo, "Time robustness in mtl and expressivity in hybrid system falsification," in *International Conference on Computer Aided Verification*. Springer, 2015, pp. 356–374.

[27] I. Haghighi, N. Mehdipour, E. Bartocci, and C. Belta, "Control from signal temporal logic specifications with smooth cumulative quantitative semantics," in *Conference on Decision and Control*. IEEE, 2019.

[28] L. Lindemann and D. V. Dimarogonas, "Robust control for signal temporal logic specifications using discrete average space robustness," *Automatica*, vol. 101, pp. 377–387, 2019.

[29] N. Mehdipour, C.-I. Vasile, and C. Belta, "Average-based robustness for continuous-time signal temporal logic," in *Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 5312–5317.

[30] Y. Gilpin, V. Kurtz, and H. Lin, "A smooth robustness measure of signal temporal logic for symbolic control," *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 241–246, 2021.

[31] A. Dokhanchi, B. Hoxha, and G. Fainekos, "On-line monitoring for temporal logic robustness," in *Intl Conf on Runtime Verification*, 2014.

[32] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*. Cambridge University Press, 1952.

[33] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, 2nd ed. Cambridge University Press, 2002.

[34] P. Varnai, "Reinforcement learning endowed robot planning under spatiotemporal logic specifications," Ph.D. dissertation, KTH, 2019.

[35] S. Sadraddini and C. Belta, "Robust temporal logic model predictive control," in *Allerton Conference on Communication, Control, and Computing*. IEEE, 2015, pp. 772–779.

[36] O. Shamir and T. Zhang, "Stochastic gradient descent for non-smooth optimization: Convergence results and optimal averaging schemes," in *International Conference on Machine Learning*, 2013, pp. 71–79.

[37] Y. Wang, W. Yin, and J. Zeng, "Global convergence of admm in nonconvex nonsmooth optimization," *Journal of Scientific Computing*, vol. 78, no. 1, pp. 29–63, 2019.

[38] F. E. Curtis and X. Que, "A quasi-newton algorithm for nonconvex, nonsmooth optimization with global convergence guarantees," *Mathematical Programming Computation*, vol. 7, no. 4, pp. 399–428, 2015.

[39] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright *et al.*, "Scipy 1.0: fundamental algorithms for scientific computing in python," *Nature methods*, vol. 17, no. 3, pp. 261–272, 2020.