SE(N) Invariance in Networked Systems

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***Abstract*— In this paper, we study the translational and rotational ($SE(N)$) invariance properties of locally interacting multi-agent systems. We focus on a class of networked systems, in which the agents have local pairwise interactions, and the overall effect of the interaction on each agent is the sum of the interactions with other agents. We show that such systems are $SE(N)$-invariant if and only if they have a special, *quasi-linear* form. The $SE(N)$-invariance property, sometimes referred to as "left invariance", is central to a large class of kinematic and robotic systems. When satisfied, it ensures independence to global reference frames. In an alternate interpretation, it allows for integration of dynamics and computation of control laws in the agents' own reference frames. Such a property is essential in a large spectrum of applications, e.g., navigation in GPS-denied environments. Because of the simplicity of the quasi-linear form, this result can impact ongoing research on design of local interaction laws. It also gives a quick test to check if a given networked system is $SE(N)$-invariant.**

I. INTRODUCTION

In this paper we present necessary and sufficient conditions for a multi-agent system with pairwise interactions to be invariant under translations and rotations of the inertial frame in which the dynamics are expressed (i.e. $SE(N)$ invariant). This kind of invariance is important because it allows agents to execute their control laws in their body reference frame [3], [2], [4], using information measured in their body reference frame, without effecting the global evolution of the system. This is critical for any scenario where global information about an agent's reference frame is not readily available, for example coordinating agents underground, underwater, inside of buildings, in space, or in any GPS denied environment [8], [1], [13].

We assume that the agents are kinematic in N-dimensional Euclidean space, and their control laws are computed as the sum over all neighbors of pairwise interactions with the neighbors. We prove that the dynamics are $SE(N)$ invariant if and only if the pairwise interactions are *quasi-linear*, meaning linear in the difference between the states of the two agents, multiplied by a nonlinear scalar gain. This result can be used as a test (does a given multi-agent controller require global information?), or as a design specification (a multi-agent controller is required that uses only local information, hence only quasi-linear pairwise interactions can be considered). It can also be used to test hypothesis about local interaction laws in biological (e.g., locally interacting cells) and physical systems.

We prove the result for agents embedded in Euclidean space of any dimension, and the result holds for arbitrary graph topologies, including directed or undirected, switching, time varying, and connected or unconnected. We show that many existing multi-agent protocols are quasi-linear and thus $SE(N)$ invariant. Examples include the interactions from the classical n-body problem [12] and most of the existing multi-agent consensus and formation control algorithms, e.g., [16], [10], [11], [18], [9], [5], [7]. In particular, explicit consensus algorithms implemented using local information in the agents' body frames [13] satisfy the $SE(N)$ invariance property, as expected. We also show that some multi-agent interaction algorithms, such as [6], are not $SE(N)$ invariant, and therefore cannot be implemented locally in practice. Finally, we consider a sub-class of quasi-linear (and therefore $SE(N)$ invariant) pairwise interaction systems, and show that they reach a consensus, using the graph Laplacian to represent the system dynamics and the typical LaSalle's invariance analysis to show convergence.

With a few exceptions [15], [17], [14], [13], the problem of invariance to global reference frames was overlooked in the multi-agent control and consensus literature. In [15], the authors discuss invariance for the particular cases of $SE(2)$ and $SE(3)$ actions, and with particular focus on virtual structures. Rotational and translational invariance are also discussed for a particular class of algorithms driving a team of agents to a rigid structure in [17]. The problem of invariance to group actions in multi-agent systems was very recently studied in [14], where the authors present a general framework to find all symmetries in a given second-order planar system. The authors' main motivation is to determine change of coordinates transformations that align the system with the symmetry directions and thus aid in stability analysis using LaSalle's principle. This paper is complementary to our work, in the sense that the authors start from a system and find invariants, while in our case we start from an invariance property and find all systems satisfying it. Our results hold for any (finite) dimensional agent state space. Finally, our characterization of invariance is algebraic, and as a result does not require any smoothness assumptions on the functions modeling the interactions. As a result, it can be used for a large class of systems, including discrete-time systems.

The rest of the paper is organized as follows. Section II defines necessary concepts and states the main result. The main result is proved in Sections III, IV, and V. Section VI considers convergence to consensus in a special class of

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systems. Section VII analyzes the $SE(N)$ invariance of several well-known systems, and conclusions are given in Section VIII.

II. Definitions and Main Result

In this section, we introduce the notions and definitions used throughout the paper. The main result of the paper is stated at the end of the section.

For a set S, we use $|S|$ to denote its cardinality. The canonical basis for the Euclidean space of dimension N, denoted by $\mathbb{R}^N$, is $e_1, \ldots, e_N$. We use I_N to denote the $N \times N$ identity matrix. The special orthogonal group acting on $\mathbb{R}^N$ is denoted by $SO(N)$. Throughout the paper, the norm $\|\cdot\|$ refers to the Euclidean norm.

Given a directed graph G, we use $V(G)$ and $E(G) \subseteq V(G) \times V(G)$ to denote its sets of nodes and edges, respectively. Given a node $i \in V(G)$, N_i stands for the set of incoming neighbors of i, i.e. $N_i = \{j \in V(G) | (i,j) \in E(G)\}$.

Let $T = \{f : \mathbb{R}^N \to \mathbb{R}^N\}$ be the set of all transformations acting on $\mathbb{R}^N$. T is a monoid with respect to function composition and is called the *transformation monoid*. Let A be a sub-semigroup of T. The *centralizer* (or *commuter*) of A with respect to T is denoted by $C_T(A)$ and is the set of all elements of T that commute with all elements of A, i.e. $C_T(A) = \{f \in T | fg = gf, \forall g \in A\}$. The centralizer $C_T(A)$ is a submonoid of T and can be interpreted as the set of transformations invariant with respect to all transformations in A. In other words, action of $f \in C_T(A)$ on $\mathbb{R}^N$ and then transformed by $g \in A$ is the same as the action of f on the transformed space $g(\mathbb{R}^N)$.

A function $f : \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to \mathbb{R}^N$ is said to be $SE(N)$ invariant if for all $R \in SO(N)$ and all $w \in \mathbb{R}^N$ the following condition holds:

$$Rf(x_1, \ldots, x_p) = f(Rx_1 + w, \ldots, Rx_p + w) \qquad (1)$$

Definition 2.1 (Pairwise Interaction System): A continuous-time pairwise interaction system is a double (G, F), where G is a graph and $F = \{f_{ij} \mid f_{ij} : \mathbb{R}^N \to \mathbb{R}^N, (i,j) \in E(G)\}$ is a set of functions associated to its edges. Each $i \in V(G)$ labels an agent, and a directed edge (i,j) indicates that node i receives information from node j. The dynamics of each agent are described by

$$\dot{x}_i = \sum_{j \in N_i} f_{ij}(x_i, x_j) \qquad (2)$$

where f_{ij} defines the influence (interaction) of j on i. For each agent $i \in V(G)$, we use $S_i(x_1, \ldots, x_{|V(G)|}) = \sum_{j \in N_i} f_{ij}(x_i, x_j)$ to denote the *total interaction* on agent i.

Remark 2.2: A discrete-time pairwise interaction system can be defined similarly by replacing differentiation ($\dot{x}_i$) with one-step difference ($\Delta x_i(k) = x_i(k+1) - x_i(k)$) in the formula above. Note that we do not assume any symmetry properties of the interaction functions f_{ij}, i.e., f_{ij} may be different from f_{ji}. Also, the graph representing the communication topology of the multi-agent system can be switching (i.e. state-dependent, or time-varying) and need not be connected.

Definition 2.3 ($SE(N)$ Invariance): A pairwise interaction system (G, F) is said to be $SE(N)$ invariant if, for all $i \in V(G)$, the total interaction functions S_i are $SE(N)$ invariant.

Definition 2.4 (Quasi-linear): A function $f : \mathbb{R}^N \to \mathbb{R}^N$ is said to be quasi-linear if there is a function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $f(x) = k(\|x\|)x$, for all $x \in \mathbb{R}^{\mathbb{N}}$. A pairwise interaction system (G, F) is said to be quasi-linear if the total interaction S_i of each agent i is a sum of quasi-linear functions. Formally, for all $i \in V(G)$:

$$\begin{aligned} S_i &= \sum_{j \in N_i} k_{ij}(\|x_j - x_i\|)(x_j - x_i), \text{ for } N \geq 3, \qquad (3)\\ S_i &= \sum_{j \in N_i} \Big(k_{ij}^{(1)}(\|x_j - x_i\|) I_2 \\ &\qquad + k_{ij}^{(2)}(\|x_j - x_i\|) J_2 \Big) (x_j - x_i), \text{ for } N = 2, \end{aligned}$$

where $k_{ij}, k_{ij}^{(1)}, k_{ij}^{(2)} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for all $j \in N_i$ are scalar gain functions and $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The difference between the cases $N = 2$ and $N \geq 3$ arises from the fact that $SO(2)$, the group of planar rotations, is Abelian, while $SO(N)$ for $N \geq 3$ is not (i.e., rotation matrices in 3 or more dimensions do not, in general, commute). To simplify the notation, we will denote the gain matrix $k_{ij}^{(1)}(\cdot) I_2 + k_{ij}^{(2)}(\cdot) J_2 = \begin{bmatrix} k_{ij}^{(1)}(\cdot) & k_{ij}^{(2)}(\cdot) \\ -k_{ij}^{(2)}(\cdot) & k_{ij}^{(1)}(\cdot) \end{bmatrix}$ also by $k_{ij}(\cdot)$. In most cases, this convention will not pose any problems because the gain matrix commutes with $SO(2)$. When it is necessary, we will consider the case $N = 2$ separately.

Note that the set of all quasi-linear functions is a submonoid of T, which will be denoted by $QL(N)$.

The main result of this paper can be simply stated as follows:

Theorem 2.5: A pairwise interaction system is $SE(N)$ invariant if and only if it is quasi-linear.

The main ingredient of the proof is that the centralizer of $SO(N)$ with respect to T is the set of all quasi-linear functions $QL(N)$. The proof, which uses an induction argument over N, is provided in Sec. IV . Building on this result, we proceed to show in Sec. V that the local interaction functions which arise from $SE(N)$ invariant pairwise interaction systems have a special form, e.g. these functions are quasi-linear functions plus some affine terms, and the sum of all the affine terms is zero. This in turn implies the main result.

III. Re-casting the Main Result

This section rephrases the main result of the paper (Theorem 2.5) in terms of total interaction functions, independent of a notion of dynamics, which has two benefits: (1) it greatly expands the applicability of the result to other cases, and (2) we do not need to assume any smoothness conditions on the functions, such as continuity or differentiability.

Theorem 3.1: Let $S(x_1,\ldots,x_p) = \sum_{j=1,j\neq i}^{p} h_j(x_i,x_j)$, where $h_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and $p \geq 1$. Then S is $SE(N)$ invariant if and only if it is the sum of quasi-linear functions in $x_j - x_i$, $j \in \{1,\ldots,p\}$, $j \neq i$, that is $S = \sum_{j=1,j\neq i}^{p} k_j(\|x_j - x_i\|)(x_j - x_i)$, where $k_j : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Proof: Let $S(x_1,\ldots,x_p) = \sum_{j=1,j\neq i}^{p} h_j(x_i,x_j)$ be an $SE(N)$ invariant function, it follows from Lemma 5.3 (which is stated and proved below) that there exists $k_j(\cdot)$ for all $j \in \{1,\ldots,p\}$, $j \neq i$, such that

$$\begin{aligned} S &= \sum_{j=1,j\neq i}^{p} (h_j(x_i,x_i) + k_j(\|x_j - x_i\|)(x_j - x_i)) \\ &= \sum_{j=1,j\neq i}^{p} h_j(x_i,x_i) + \sum_{j=1,j\neq i}^{p} k_j(\|x_j - x_i\|)(x_j - x_i) \\ &= \sum_{j=1,j\neq i}^{p} k_j(\|x_j - x_i\|)(x_j - x_i) \end{aligned}$$

The proof is now complete. ∎

Remark 3.2: Notice that Thm. 3.1 is a statement about the total interaction functions and not about each local interaction functions, e.g. it does not say that the local interactions functions $h_j(x_i,x_j)$ are quasi-linear in $x_j - x_i$. This fact is shown explicitly in Lemma 5.3 which also proves that all affine terms $h_j(x_i,x_i)$ cancel each other out.

Thm. 2.5 follows immediately from Thm. 3.1, since we can apply Thm. 3.1 on the total interaction function S_i of any agents i.

Remark 3.3: Notice that the presented result does not concern the system stability or asymptotic properties of the system trajectories, since $SE(N)$ invariance is a property essentially about reference frames. Also, because of the same considerations, our results hold true for the case of switching network topologies. We investigate the stability of a subclass of these systems in Sec. VI, which we prove converge to a consensus state.

Remark 3.4: Lastly, we want to draw attention to the fact that the result holds because the total interaction functions are the sum of local pairwise interactions between neighboring agents. In case of general total interaction functions which do not have this additive form, the result does not hold. As a counter-example, consider the total interaction function $S(x_1,x_2,x_3) = \|x_2 - x_1\|(x_3 - x_2)$. We can easily see that S is $SE(N)$ invariant, but is not a sum of quasi-linear functions.

IV. Characterizing the Centralizers of $SO(N)$

In this section, we prove that functions which commute with $SO(N)$ are quasi-linear. First, we establish the claim for $N = 2$, which is treated separately. Some of the proofs in this section are omitted due to brevity.

Proposition 4.1: The centralizer of $SO(2)$ with respect to T is the submonoid $\{(k_1(\|x\|)I_2 + k_2(\|x\|)J_2)\,x\}$, where $k_1, k_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Before we proceed with the case $N \geq 3$, we provide two lemma that are used in subsequent proofs. The following lemma shows the intuitive fact that the only vector invariant under all rotations is the null vector.

Lemma 4.2: Let $x \in \mathbb{R}^N$. If $Rx = x$ for all $R \in SO(N)$, $N \geq 2$, then $x = 0$.

Lemma 4.3: Let $f = (f_1,\ldots,f_N) : \mathbb{R}^N \to \mathbb{R}^N$ such that f commutes with all elements of $SO(N)$, then $x^T f(x) = \|x\|\, f_1(\|x\|\, e_1)$.

Proof: Let $x \in \mathbb{R}^N$ and $R \in SO(N)$ such that $Rx = \|x\|\, e_1$ or equivalently $x = R^T \|x\|\, e_1$. It follows that $f(x) = f(R^T \|x\|\, e_1) = R^T f(\|x\|\, e_1)$. Finally, $x^T f(x) = x^T R^T f(\|x\|\, e_1) = (Rx)^T f(\|x\|\, e_1) = \|x\|\, e_1 f(\|x\|\, e_1) = \|x\|\, f_1(\|x\|\, e_1)$. ∎

The following three lemma establish the case $N = 3$ which forms the base case of the induction argument used in the proof of Thm. 4.7.

Lemma 4.4: Let $u = (u_1,u_2,u_3) \in \mathbb{R}^3$ such that $\|u\| = 1$ and $u \neq \pm e_1$, then $R_u = \begin{bmatrix} u_1 & 0 & -\sqrt{u_2^2 + u_3^2} \\ u_2 & \frac{u_3}{\sqrt{u_2^2+u_3^2}} & \frac{u_1 u_2}{\sqrt{u_2^2+u_3^2}} \\ u_3 & -\frac{u_2}{\sqrt{u_2^2+u_3^2}} & \frac{u_1 u_3}{\sqrt{u_2^2+u_3^2}} \end{bmatrix}$ is a proper rotation matrix in $SO(3)$.

Proof: It follows from the definition of $SO(3)$. ∎

Lemma 4.5: Let $f = (f_1,f_2,f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ such that f commutes with all elements of $SO(3)$, then

$$f_1(x) = -f_1(-x_1,-x_2,x_3) \tag{4}$$
$$f_1(x) = -f_1(-x_1,x_2,-x_3) \tag{5}$$
$$f_2(x) = f_1(x_2,-x_1,x_3) \tag{6}$$
$$f_3(x) = f_1(x_3,x_2,-x_1) \tag{7}$$

where $x = (x_1,x_2,x_3) \in \mathbb{R}^3$.

Proof: The above relationships can be obtained using 90° rotation matrices around the axes e_1, e_2 and e_3. ∎

Proposition 4.6: The centralizer of $SO(3)$ with respect to T is the monoid of quasi-linear functions $QL(3)$.

Proof: Let $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ such that $x \neq \alpha e_1$, $\alpha \in \mathbb{R}$ and $f = (f_1,f_2,f_3) : \mathbb{R}^3 \to \mathbb{R}^3$. Let $u = \frac{x}{\|x\|}$ and R_u as in Lemma 4.4, we have $x = R_u \|x\|\, e_1$ and $u_i = \frac{x_i}{\|x\|}$. Using the commutation property we obtain $f(x) = f(R_u \|x\|\, e_1) = R_u f(\|x\|\, e_1)$ and writing the equation for f_1, it follows that

$$f_1(x) = u_1 f_1(\|x\|\, e_1) - \sqrt{u_2^2 + u_3^2}\, f_3(\|x\|\, e_1) \tag{8}$$

Using the equality from Lemma 4.5, Eq. (7), we have $f_3(\|x\|,0,0) = f_1(0,0,-\|x\|)$. On the other hand, it follows from Eq. (4) that $f_1(0,0,\alpha) = -f_1(0,0,\alpha)$, which implies $f_1(0,0,\alpha) = 0$ for all $\alpha \in \mathbb{R}$. It follows that $f_3(\|x\|\, e_1) = 0$ for all $x \in \mathbb{R}^3$, $x \neq \alpha e_1$ and $\alpha \in \mathbb{R}$. Thus, Eq. (8) can be simplified to

$$f_1(x) = x_1 \frac{f_1(\|x\|\, e_1)}{\|x\|} = x_1 k(\|x\|) \tag{9}$$

where $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is $k(\alpha) \triangleq \frac{f_1(\alpha e_1)}{\alpha}$, $\alpha \geq 0$. The general form of $f(x) = k(\|x\|)x$ is obtained using Eq. (6) and (7).

The case $x=0$ follows easily from Lemma 4.2, because it implies $f(0)=0$. The remaining case $x=\alpha e_1$ is trivial; $f(\alpha e_1)=[f_1(\alpha e1)\ f_2(\alpha e1)\ f_3(\alpha e1)]^T=[\alpha\frac{f_1(\alpha e_1)}{\alpha}\ 0\ 0]^T=k(\|x\|)x$, where $f_2(\alpha e_1)=0$ and $f_3(\alpha e_1)=0$ follow from Eq. (6), (5) and Eq. (7), (4), respectively. ■

Theorem 4.7: The centralizer of $SO(N)$ with respect to T is the monoid of quasi-linear functions $QL(N)$, for all $N\geq 3$.

Proof: The proof follows from an induction argument with respect to N. The base case is established by Prop. 4.6. To simplify the notation, given a vector $x=(x_1,\ldots,x_N)$ we will denote by $x_{i:j}$, $i<j$, the sliced vector $(x_i,\ldots,x_j)\in\mathbb{R}^{j-i+1}$.

The induction step: Let $x\in\mathbb{R}^{N+1}$, $x\neq 0$, and $R_1=\begin{bmatrix}R & 0\\ 0 & 1\end{bmatrix}$, $R_2=\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}$, where $R\in SO(N)$. Using R_1, it follows that $Rf_{1:N}(x_{1:N},x_{N+1})=f_{1:N}(Rx_{1:N},x_{N+1})$. Applying the induction hypothesis, we obtain

$$f_{1:N}(x_{1:N},x_{N+1})=k_1(\|x_{1:N}\|,x_{N+1})x_{1:N} \tag{10}$$

Similarly, using R_2 we have $Rf_{2:N+1}(x_1,x_{2:N+1})=f_{2:N+1}(x_1,Rx_{2:N+1})$ and obtain

$$f_{2:N+1}(x_1,x_{2:N+1})=k_2(\|x_{2:N+1}\|,x_1)x_{2:N+1} \tag{11}$$

Equating Eq. (10) and (11) for f_2 and assuming w.l.o.g. $x_2\neq 0$, we get a constraint between the two gains

$$k_2(\|x_{2:N+1}\|,x_1)=k_1(\|x_{1:N}\|,x_{N+1}) \tag{12}$$

Thus, we obtain f_{N+1} in terms of the gain k_1 by using the last equality from Eq. (11) and Eq. (12) to substitute k_2 for k_1

$$f_{N+1}(x_1,\ldots,x_{N+1})=k_1(\|x_{1:N}\|,x_{N+1})x_{N+1} \tag{13}$$

Finally, putting all the components of f obtained from Eq. (10) and (13) together and left multiplying it by x^T, we get

$$\begin{aligned}x^Tf(x)&=\sum_{i=1}^{N+1}x_i^2k_1(\|x_{1:N}\|,x_{N+1})\\&=\|x\|^2k_1(\|x_{1:N}\|,x_{N+1})=\|x\|f_1(\|x\|e_1)\end{aligned}$$

where the last equality follows from Lemma 4.3. It follows that $k_1(\|x_{1:N}\|,x_{N+1})=\frac{f_1(\|x\|e_1)}{\|x\|}\triangleq k(\|x\|)$. Thus, $f(x)=k(\|x\|)x$, which concludes the proof. ■

V. $SE(N)$ INVARIANT FUNCTIONS

In this section, we use the result from the previous section that $C_T(SO(N))=QL(N)$ in order to characterize $SE(N)$ invariant functions which arise from pairwise interaction systems.

Proposition 5.1: A function $h(x_1,x_2):\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is $SE(N)$ invariant if and only if h is quasi-linear in x_2-x_1.

Proof: Trivially, a quasi-linear function $h(x_1,x_2)=k(\|x_2-x_1\|)(x_2-x_1)$ is $SE(N)$ invariant. Conversely, if $R=I_N$ and $w=-x_2$, then $h(x_1,x_2)=h(x_1-x_2,x_1-x_2)=h(x_1-x_2,0)\triangleq\hat{h}(x_2-x_1)$. Let $x\in\mathbb{R}^N$ and $R\in SO(N)$, it follows that $R\hat{h}(x)=Rh(-x,0)=h(-Rx,0)=\hat{h}(Rx)$. Since $\hat{h}$ commutes with all elements of $SO(N)$ it follows that it is quasi-linear. Thus we have $h(x_1,x_2)=\hat{h}(x_2-x_1)=k(\|x_2-x_1\|)(x_2-x_1)$. ■

Lemma 5.2: Let $h_1,h_2:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$, then $S(x_0,x_1,x_2)=h_1(x_0,x_1)+h_2(x_0,x_2)$ is an $SE(N)$ invariant function if and only if there exists $k_1(\cdot)$ and $k_2(\cdot)$ such that for all $x_0,x_1,x_2\in\mathbb{R}^N$ we have

$$h_1(x_0,x_1)=h_1(x_0,x_0)+k_1(\|x_1-x_0\|)(x_1-x_0) \tag{14}$$
$$h_2(x_0,x_2)=h_2(x_0,x_0)+k_2(\|x_2-x_0\|)(x_2-x_0) \tag{15}$$

and $h_1(x_0,x_0)+h_2(x_0,x_0)=0$.

Proof: It is easy to show that if S is the sum of functions satisfying Eq. (14), (15) and the zero-sum constraint, then S is $SE(N)$ invariant. Conversely, let $f_1(a,b)=h_1(a,b)+h_2(a,a)$ and $f_2(a,b)=h_1(a,a)+h_2(a,b)$, where $f_1,f_2:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ and $a,b\in\mathbb{R}^N$. It follows immediately that f_1 and f_2 are $SE(N)$ invariant, because $h_1(x_0,x_1)+h_2(x_0,x_2)$ is $SE(N)$ invariant. Therefore, we have by Prop. 5.1 that $f_1(a,b)=k_1(\|b-a\|)(b-a)$ and $f_2(a,b)=k_2(\|b-a\|)(b-a)$. Choosing $a=b$ in any of the previous two equations, we obtain $h_1(a,a)+h_2(a,a)=0$. Finally, we obtain $h_1(a,b)=-h_2(a,a)+f_1(a,b)=h_1(a,a)+k_1(\|b-a\|)(b-a)$ and $h_2(a,b)=-h_1(a,a)+f_2(a,b)=h_2(a,a)+k_2(\|b-a\|)(b-a)$. ■

Lemma 5.3: Let $h_1,\ldots,h_p:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$, $p\in\mathbb{Z}_{\geq 2}$, then $S(x_0,\ldots,x_p)=\sum_{i=1}^p h_i(x_0,x_i)$ is an $SE(N)$ invariant function if and only if there exists $k_i(\cdot)$, $i\in\{1,\ldots p\}$, such that for all $x_0,x_1,\ldots,x_p\in\mathbb{R}^N$ we have

$$h_i(x_0,x_i)=h_i(x_0,x_0)+k_i(\|x_i-x_0\|)(x_i-x_0) \tag{16}$$

for all $i\in\{1,\ldots,p\}$ and

$$\sum_{i=1}^p h_i(x_0,x_0)=0 \tag{17}$$

Proof: We will prove the lemma by induction with respect to p. The base step $p=2$ is established by Lemma 5.2. For the induction step, we assume that Lemma 5.3 holds for p and we must show that it also holds for $p+1$. As before, quasi-linearity trivially implies $SE(N)$ invariance.

Let $x_{p+1}=x_1$ and define the function $h_1'(x_0,x_1)=h_1(x_0,x_1)+h_{p+1}(x_0,x_1)$. Clearly $h_1'(x_0,x_1)+\sum_{i=2}^p h_i(x_0,x_i)$ is an $SE(N)$ invariant function and by the induction hypothesis we have for all $i\in\{2,\ldots,p\}$

$$\begin{aligned}h_i(x_0,x_i)&=h_i(x_0,x_0)+k_i(\|x_i-x_0\|)(x_i-x_0)\\h_1'(x_0,x_1)&=h_1'(x_0,x_0)+k_1'(\|x_1-x_0\|)(x_1-x_0)\\&=h_1(x_0,x_0)+h_{p+1}(x_0,x_0)\\&\quad+k_1'(\|x_1-x_0\|)(x_1-x_0)\end{aligned}$$

and $h_1'(x_0,x_0)+\sum_{i=2}^p h_i(x_0,x_0)=\sum_{i=1}^{p+1}h_i(x_0,x_0)=0$.

Similarly, let $x_{p+1}=x_2$ and define $h_2'(x_0,x_2)=h_2(x_0,x_2)+h_{p+1}(x_0,x_2)$. Using the same argument as before, we obtain $h_1(x_0,x_1)=h_1(x_0,x_0)+$

$k_1(\|x_1 - x_0\|)(x_1 - x_0)$. Substituting h_1 in the expression of h_1' and solving for h_{p+1} we have

$$\begin{aligned} h_{p+1}(x_0, x_{p+1}) &= h_1'(x_0, x_{p+1}) - h_1(x_0, x_{p+1}) \\ &= h_{p+1}(x_0, x_0) + k_{p+1}(\|x_{p+1} - x_0\|) \end{aligned}$$

This concludes the proof. ∎

VI. STABILITY OF $SE(N)$ INVARIANT SYSTEMS

In this section, we study the stability of $SE(N)$ invariant pairwise interaction systems, showing that a certain subclass of such systems converge to a consensus state (one in which all agents' states are equal). Because of Thm. 2.5, the stability of such systems can be reduced to a problem involving only scalar gain functions. In the following, we provide a sufficient condition for stability by assuming some additional constraints on the scalar gains.

Let (G, F) be a continuous-time pairwise interaction system, where each $f_{ij} \in F$ is quasi-linear. Let $n = |V(G)|$ and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be the stacked state vector. Using this notation, the system dynamics may be written in the following form:

$$\dot{x} = -(L \otimes I_N)x$$

where $\otimes$ denotes the Kronecker product and L is the $n \times n$ weighted Laplacian matrix of G, i.e. for all $i, j \in \{1, \ldots, n\}$

$$L_{ij} = \begin{cases} \sum_{p \in N_i} k_{ip}(\|x_i - x_p\|) & \text{for } i = j \\ -k_{ij}(\|x_i - x_j\|) & \text{for } i \neq j \text{ and } (i,j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Also, define the set of a consensus states as follows

Definition 6.1: The states in the set

$$\Omega = \{x | x_i = x_j, \forall i, j \in V(G)\}$$

are called consensus states, and if $x(t) \to \Omega$ as $t \to \infty$ we say the system converges to a consensus.

Theorem 6.2: Let (G, F) be a continuous pairwise interaction system such that G is connected and time-invariant, and each $f_{ij} \in F$ is quasi-linear. If $k_{ij}(\alpha) = k_{ji}(\alpha)$, $k_{ij}(\alpha) > 0$, $\alpha \geq 0$, and k_{ij} are continuous, then x converges to a point in $\{x | x_i = x_j, \forall i, j \in V(G)\}$.

Proof: Consider the Lyapunov function $V(x) = \frac{1}{2}\sum_{(i,j)\in E(G)} \|x_i - x_j\|^2 + \bar{x}^T\bar{x}$, where $\bar{x} = \frac{1}{n}\sum_{k=1}^n x_k$. By construction, V is positive definite. Next we show that it is radially unbounded. Let $x \in \mathbb{R}^{Nn}$ be such that $\|x\| \to \infty$. There exists an agent $i \in \{1, \ldots, n\}$ such that $\|x_i\| \to \infty$, $x_i \in \mathbb{R}^N$, and we have the following inequality

$$\begin{aligned} \|x_i\| = \|x_i - \bar{x} + \bar{x}\| &= \left\| \frac{1}{n}\sum_{j=1}^n (x_i - x_j) + \bar{x} \right\| \\ &\leq \frac{1}{n}\sum_{j=1}^n \|x_i - x_j\| + \|\bar{x}\| \end{aligned}$$

The inequality shows that either $\|\bar{x}\| \to \infty$, in which case $V(x) \to \infty$, or there exists i, j such that $\|x_i - x_j\| \to \infty$. In case i and j are adjacent, $(i,j) \in E(G)$, then $V(x) \to \infty$. Otherwise, there exists a path $k_1, \ldots, k_m$, with $k_1 = i$ and $k_m = j$, because G is assumed to be connected. Thus, we have

$$\|x_i - x_j\| = \left\| \sum_{l=1}^m x_{k_l} - x_{k_{l+1}} \right\| \leq \sum_{l=1}^m \left\| x_{k_l} - x_{k_{l+1}} \right\|$$

which shows that there exists $(k_l, k_{l+1}) \in E(G)$ such that $\left\| x_{k_l} - x_{k_{l+1}} \right\| \to \infty$, which in turn implies that $V(x) \to \infty$.

Next, we proceed to compute the derivative of $\bar{x}$

$$\begin{aligned} \dot{\bar{x}} &= \frac{1}{n}\sum_{i=1}^n \dot{x}_i = \frac{1}{n}\sum_{i=1}^n \sum_{j \in N_i} k_{ij}(\|x_j - x_i\|)(x_j - x_i) \\ &= \frac{1}{n}\sum_{(i,j)\in E(G)} (k_{ij}(\|x_i - x_j\|) - k_{ji}(\|x_i - x_j\|))(x_j - x_i) \\ &= 0 \end{aligned}$$

We are now ready to compute the total derivative of V,

$$\begin{aligned} &\dot{V}(x) = \\ &= \sum_{p=1}^n \frac{\partial V}{\partial x_p}^T \dot{x}_p = \frac{1}{2}\sum_{p=1}^n \left(\frac{\partial}{\partial x_p} \left(\sum_{(i,j)\in E(G)} \|x_i - x_j\|^2 \right) \right)^T \dot{x}_p \\ &= \frac{1}{2}\sum_{p=1}^n \left(\sum_{j \in N_p} 2(x_p - x_j) \right)^T \left(\sum_{j \in N_p} k_{pj}(\|x_p - x_j\|)(x_j - x_p) \right) \\ &= -x^T(L \otimes I_N)x \end{aligned}$$

Because $k_{ij}(\cdot)$'s are always positive, it follows that L is diagonally dominant and thus positive semi-definite. This in turn implies that $\dot{V}$ is negative semi-definite. The solutions of the equation $\dot{V}(x) = -x^T(L \otimes I_N)x = 0$ are also the equilibrium points of the system, i.e. they satisfy $(L \otimes I_N)x = 0$. Thus, by LaSalle's invariance principle x converges to the invariant set $\{x | \dot{x} = 0\}$, which in this case is $\Omega = \{x | x_i = x_j, \forall i, j \in V(G)\}$, hence the system converges to a consensus. ∎

The result can be extended straightforwardly to switching graph topologies using known methods, e.g., [16]. We conjecture that an $SE(N)$ invariant pairwise multi-agent system reaches a consensus if and only if it takes on this form, although the proof is left for future work.

VII. ANALYSIS OF EXISTING CONTROLLERS

In this section we consider several existing pairwise multi-agent systems that have been studied in the literature. We show that many of these are $SE(N)$ invariant, although we also show an example that is not, and one that is only $SE(N)$ invariant under certain conditions. These results are summarized in Table I.

The first example in Tab. I was proposed in [9] to model swarm aggregation and is a quasi-linear system because $g(\cdot)$ is a quasi-linear function. The second [6], third [11] and fourth [18] examples define the agents' dynamics based on potential functions. The example from [6] is not quasi-linear, because the potential function whose gradient is used for navigation depends explicitly on the agents' states, as opposed to distances between agents' states, and thus its gradient cannot be a quasi-linear function. We can conclude that the multi-agent system in the second example is not $SE(N)$ invariant. On the other hand, the example from [18]

TABLE I: The table contains examples of networked systems that are quasi-linear, except for the second example and possibly the fourth. It follows that the quasi-linear systems below are also $SE(N)$ invariant by Thm. 2.5. All systems have n agents and for each agent $i \in \{1,\dots,n\}$, we denote by x_i its state. The maps $V_I, V_h, V_{ij} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ represent potential functions. In the example from [11], $\widetilde{x}_p$ represents the state of a virtual leader $p \in \{1,\dots,m\}$.

System dynamics	Reference	Quasi-linear?
$\dot{x}_i = \sum_{j=1}^{n} g(x_i - x_j)$, $g(y) = -y\left(a - b\exp\left(-\frac{\|y\|^2}{c}\right)\right)$	[9]	Yes
$\dot{x}_i = -\alpha \nabla_{x_i}\left(\frac{\gamma_i(x)}{(\gamma_i(x)^k + \beta_i(x))^{1/k}}\right)$	[6]	No
$\ddot{x}_i = -\sum_{j\neq i}^{n} \nabla_{x_i} V_I(\|x_i - x_j\|) - \sum_{k=0}^{m-1} \nabla_{x_i} V_h(\|x_i - \widetilde{x}_k\|), 1 \leq i \leq n$, $\ddot{\widetilde{x}}_p = \widetilde{f}_p(x_j, \widetilde{x}_k, \dot{x}_j, \dot{\widetilde{x}}_k), 0 \leq p \leq m$	[11]	Yes or no.
$\ddot{x}_i = -\sum_{j\in N_i} \nabla_{x_i} V_{ij}(\|x_i - x_j\|) - \sum_{j\in N_i}(\dot{x}_i - \dot{x}_j)$	[18]	Yes.
$\dot{x}_i = u_i$ or $x_i(k+1) = x_i(k) + u_i$, $u_i = \sum_{j\in N_i} a_{ij}(x_i - x_j),\ a_{ij} \in \mathbb{R}$ or $u_i = \sum_{j\in N_i} (\|x_i - x_j\|^2 - d_{ij})(x_i - x_j),\ d_{ij} \in \mathbb{R}$	[16], [10], [5]	Yes
$\ddot{x}_i = \frac{1}{m_i}\sum_{j=1, j\neq i}^{n} \frac{G m_i m_j}{\|x_i - x_j\|^3}(x_j - x_i)$	[12]	Yes

is quasi-linear, because the gradients of $V_{ij}(\cdot)$ are quasi-linear functions. In the fourth example, the system is quasi-linear if and only if the dynamics of the virtual leaders $\widetilde{f}_p$ are sums of quasi-linear functions, $1 \leq p \leq m$. In the fifth example, we have consensus and formation control protocols from [16], [10] and [5]. It is easy to see that these systems are quasi-linear. The last example shows a system of n point masses which interact with each other due to gravity. This system is also quasi-linear and thus exhibits $SE(N)$ invariance, a fact which is well known in Hamiltonian mechanics [12].

VIII. Conclusions

In this paper we present the notion of $SE(N)$ invariance, invariance to translations and rotations, in the case of multi-agent systems. Systems which have this property are independent of global reference frames, which means they can be implemented using information measured in each agent's body reference frame, and executed in each agent's body reference frame, without effecting the global evolution of the system. This is of critical importance in the context of applications where information about a global reference frame cannot be obtained, for example in GPS-denied environments. The main contribution of the paper is to fully characterize pairwise interaction systems that are $SE(N)$ invariant. We show that pairwise interaction systems are $SE(N)$ invariant if and only if they have a special *quasi-linear* form. Because of the simplicity of the quasi-linear form, this result can impact ongoing research on design of local interaction laws. The result can also be used to quickly check if a given networked system is $SE(N)$ invariant. Lastly, we describe a subset of $SE(N)$ invariant pairwise interaction systems that reach a consensus. We prove this result by exploiting their quasi-linear structure.

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